The AdS-CFT Correspondence: *A Review*

Paul Plant, Quantum Fields and Fundamental Forces MSc,

September 2014, Imperial College London

Supervisor: Professor Daniel J. Waldram

Submitted in partial fulfilment of the requirements for the degree of Master of Science of Imperial College London

Contents

To Vlad

Abstract

This paper reviews the AdS/CFT Correspondence, introduced by Juan Malcadena in 1997, relating Type IIB string theory on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ Super Yang-Mills theory in four dimenstions. We provide a comprehensive study of the fundamental aspects of the conjecture, including: supersymmetry, conformal symmetry, geometry of AdS space, and string theory. By looking at a configuration of parallel D3-branes we are able to realise the correspondence and relate the dynamics of the bulk string theory to two- and three-point correlation functions within Super Yang-Mills theory.

1 Introduction

When string theory was introduced in the 1960s as a theory to describe the strong force, it was plagued by serious pitfalls, and had to be abandoned in favour of the formidable QCD. However, one of the problems, a masless spin-two particle, revealed itself to be the graviton and allowed the theory to rise from the ashes as a consistent description of quantum gravity. It was not until thirty years later that its intentions have come full-circle - string theory and the strong force appear to be dual. In 1997, this conjecture was brought to the world's attention by Malcadena[16]. Focussing on extended objects within string theory, known as D-branes, he was able to use Type IIB string theory propagating on an $AdS_5 \times S^5$ background to describe Super Yang-Mills, a conformal theory living in four-dimensional flat space. This was dubbed the *AdS/CFT Correspondence*. The correspondence focuses on the principle of holography, where the information contained within a $(d + 1)$-dimensional gravity theory can be described completely by a d-dimensional field theory living at the boundary. Through this, we can relate the observables of both theories - in particular, the fields propagating in the bulk of string theory and the correlation functions of Super Yang-Mills. The correspondence remains a conjecture as this gauge-gravity duality is an example of a strong-weak duality, where the valid regime for the gauge theory is not reliable to describe the string theory, and vice versa, rendering it very difficult to prove. Despite this, the duality allows us to perform strong coupling calculations in Super Yang-Mills that would be very difficult otherwise[1]. Although this isn't a realistic theory, the fundamental ideas are important in the understanding of strongly-coupled quantum field theories.

The correspondence draws upon many areas of physics, and this paper aims to explore each of these components in order to provide the reader with a firm background understanding and alluring relations. We begin by looking at supersymmetry, important as both Super-Yang Mills and string theory are supersymmetric. By studying the structure of the algebra and representations, we are able to find the content of the $\mathcal{N} = 4$ Super Yang-Mills theory, which is invariant under the conformal and superconformal algebras seen in Section 3. Following this, we look to the stringy side, studying the geometrical properties of AdS, before moving onto the spectrum of string theory in Section 5. Here we introduce the D-branes by looking at them as endpoints of open strings, and geometrically engineer the configuration of branes to give rise to an $SU(N)$ gauge theory on their worldvolume. The second viewpoint is introduced in Section 6, looking at the branes as solutions of ten-dimensional supergravity, whose presence deforms the geometry nearby to look like $AdS \times S^5$. Finally, everything is brought together in the form of the correspondence in section 7. By looking at the limits of validity and dynamics of string theory, we are able to derive the correlation functions of Super Yang-Mills from the fields propagating in the bulk.

2 Supersymmetry and $\mathcal{N} = 4$ Super Yang-Mills

Physicsists strive for more by looking for less, forcing separate parts to fit inside one whole. The Poincaré Group, our beloved spacetime symmetry group, was thought to be exempt from closing into an algebra alongside the internal symmetries by the Coleman-Mandula theorem, where they declared in their 1967 paper that their was no way to combine spacetime and internal symmeties in anything but a trivial way[7]. However, not content with this, Haag, Lopuszanski and Sohnius found that we could include spacetime symmetries if we allowed both commutators and anticommutators in the algebra[11]. Thus, we could make a fermionic extension to the Poincaré group known as the Super-Poincaré group, whose algebra we discuss below.

Why is supersymmetry important? It can be used as a potential answer to some of the major obstacles seen today in theoretical research:

- it can provide an answer to the heirarchy problem, where fundamental masses do not equal experimental masses
- it can be used to extend the standard model, with aim to combine $U(1) \times SU(2) \times SU(3)$ into a larger symmetry group, such as $SU(5)$
- extensions of the standard model include SUSY, such as dark matter and string theory.

One of the consquences of supersymmetry (SUSY) is an equivalence of mass between bosons and fermions. Of course, this isn't seen in nature, which means it is a symmetry that must be broken at suitably low energies. Attentions were turned to finding SUSY at the LHC, and the failure to do so yet has led to many people abandoning the theory. However, for the course of this paper we hold strong our faith. Below we discuss the SUSY algebra and its representations, leading us to $\mathcal{N} = 4$ Super Yang-Mills theory.

2.1 Supersymmetry Algebra

The SUSY algebra is a fermionic extension of the Poincaré, introducing fermionic generators Q_α and $\bar{Q}_{\dot{\alpha}}$, transforming as spinors in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations respectively. These generators act on bosons and fermions, flipping their nature, and are key within SUSY as they allow for both to be placed on a par. The symmetry group enjoys what is known as a graded Lie algebra, due to its inclusion of both bosons and fermions. We can define the algebra vector space V spanning $V = V_B \oplus V_F$, with V_B the fermionic vector space and V_F the fermionic vector space. The composition law is given as:

$$[,\} = V \times V \to V$$

$$\begin{aligned}
[V_B, V_B\} &= [V_B, V_B] \in V_B \\
[V_B, V_F\} &= [V_B, V_F] \in V_F \\
[V_F, V_F\} &= \{V_F, V_F\} \in V_B
\end{aligned}$$

This forms the super-Poincaré algebra, and it is generated by

- Lorentz Transformations $M_{\mu\nu}$
- Translations P_μ
- Supercharges $Q_\alpha, \bar{Q}_{\dot{\alpha}}$

The next step is to look at the underlying structure of the SUSY algebra. This is important as we need to see what the Casimirs of the group are, and thus how we may label and build the irreducible representations (irreps). The simplest case is $\mathcal{N} = 1$ SUSY, where we have only one copy of each supercharge. We also focus on the four-dimensional representations, as the end goal is to be able to represent four-dimensional Super Yang-Mills theory.

$[\mathbf{Q}_\alpha, \mathbf{M}_{\mu\nu}]$

The supercharges are spinors, so transform as a spinor under the Lorentz group. Infinitesimally the transformation reads

$$Q_\alpha \to Q'_\alpha = exp(-\frac{1}{2}\omega^{\mu\nu}\sigma_{\mu\nu})_\alpha^\beta Q_\beta \simeq (\mathbb{I} - \frac{i}{2}\omega^{\mu\nu}\sigma_{\mu\nu})_\alpha^\beta Q_\beta.$$

They are also operators, with transformation

$$Q'_\alpha = U^\dagger Q_\alpha U \simeq (1 + \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu})Q_\alpha(1 - \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}).$$

Comparing the RHS from both equations, we find

$$-\frac{i}{2}\omega^{\mu\nu}(\sigma_{\mu\nu})_\alpha^\beta Q_\beta = -\frac{i}{2}\omega^{\mu\nu}(Q_\alpha M_{\mu\nu} - M_{\mu\nu}Q_\alpha),$$

giving the commutator between the Lorentz generator and supercharges as $[Q_\alpha, M_{\mu\nu}] = (\sigma_{\mu\nu})_\alpha^\beta Q_\beta$. Comparing to the structure of the Lie algebra above, it's as expected: the commutation relation between a boson and fermion gives a fermion.

$[\mathbf{Q}_\alpha, \mathbf{P}_\mu]$

From the structure of the Lie algebra, the result of this commutator must be a fermionic generator. We can also look at the indices and deduce that there must be one free spinor and one Lorentz index. The only object that we have at our disposal is the Pauli matrix $\sigma^\mu_{\alpha\dot{\alpha}}$ and we make the ansatz

$$[Q_\alpha, P_\mu] = c(\sigma_\mu)_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}},$$

where the Pauli matrix changes the handedness of the spinor. The constant c is still undetermined - just as a regular algebra respects the Jacobi identity, our super-algebra respects the super-Jacobi identity[4], whose form follows from the composition law above. For the case of P^μ and Q_α:

$$[P_\mu, [Q_\alpha, P_\nu]] + [P_\nu, [P_\mu, Q_\alpha]] + [Q_\alpha, [P_\nu, P_\mu]] = 0.$$

The third term vanishes as translations commute, leaving

$$\begin{aligned} [P_\mu, c(\sigma_v)_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}] - [P_\nu, c(\sigma_\mu)_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}] &= cc * [(\sigma_\nu)_{\alpha\dot{\alpha}}(\sigma_\mu)^{\dot{\alpha}\beta}Q_\beta - (\sigma_\mu)_{\alpha\dot{\alpha}}(\sigma)^{\dot{\alpha}\beta}Q_\beta], \\ &= cc * (\sigma_\nu\sigma_\mu - \sigma_\mu\sigma_\nu)_\alpha^\beta Q_\beta, \\ &= 0. \end{aligned}$$

The term within the brackets satisfies the Dirac algebra and is non-zero, implying that the constant $c = 0$, and $[Q_\alpha, P_\mu] = 0$.

$\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\}$

Using a similar method as above, we postulate that we need two free spinor indices on the RHS. Because of the Coleman-Mandula theorem, the only generator that can be placed on the RHS is $M_{\mu\nu}$[4], so that $\{Q_\alpha, Q_\beta\} = k\sigma^{\mu\nu}_{\alpha\beta}M_{\mu\nu}$, where $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\nu)$. As translations and Lorentz transformations do not commute, by the super-Jacobi identity this anticommutator must vanish.

$\{\mathbf{Q}_\alpha, \bar{\mathbf{Q}}_{\dot{\alpha}}\}$

Recalling that the supercharges Q_α, $\bar{Q}_{\dot{\alpha}}$ transform in the left and right hand representations respectively, the product of the commutator should be in the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation, i.e. a Lorentz vector. Within our algebra, the object we have is P_μ, and again we can look at the required indices to find

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu, \tag{1}$$

with 2 being a canonical choice[19]. This is a very interesting outcome, two SUSY transformations results in a translation in spacetime. SUSY is infact a spacetime symmetry; it knows all about spacetime!

2.2 Consequences of the Supersymmetry algebra

The structure of the super-Poincaré algebra encodes three main important features of supersymmetry. These are[30]:

- Positivity of Energy
- Equal Number of Bosons and Fermions
- Equal Masses for Bosons and Fermions

2.2.1 Positivity of Energy

Beginning with the identity (1)

$$\bar{\sigma}^{\nu\alpha\dot{\beta}}\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\bar{\sigma}^{\nu\alpha\dot{\beta}}\sigma^\mu_{\alpha\dot{\beta}} P_\mu,$$

we then use the Dirac algebra to turn this into a metric

$$\bar{\sigma}^{\nu\alpha\dot{\beta}}\sigma^\mu_{\alpha\dot{\beta}} = Tr(\bar{\sigma}^\nu\sigma^\mu) = \frac{1}{2}Tr(\bar{\sigma}^\nu\sigma^\mu - \bar{\sigma}^\mu\sigma^\nu) = \frac{1}{2}Tr(\eta^{\nu\mu}\mathbb{I}) = 2\eta^{\nu\mu}.$$

Inserting this into the above equation

$$\bar{\sigma}^{\nu\alpha\dot{\beta}}\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 4\eta^{\nu\mu}P_\mu = P^\nu. \tag{2}$$

We define energy to be $P^0 = E$, and can assess the expectation value of an arbitrary state $|\psi\rangle$ to be in this energy state E. We also note that $\bar{Q}_{\dot{\alpha}} = (Q_\alpha)^\dagger$, i.e. the Hermitian conjugate.

$$\begin{aligned} \langle\psi| E |\psi\rangle &= \frac{1}{4}\sum_\alpha \left(\langle\psi| Q_\alpha Q^\dagger_\alpha |\psi\rangle + \langle\psi| Q^\dagger_\alpha Q_\alpha |\psi\rangle\right) && (3) \\ &= \frac{1}{4}\sum_\alpha \left(|\, Q^\dagger_\alpha |\psi\rangle\, |^2 + |\, Q_\alpha |\psi\rangle\, |^2\right) \geq 0, && (4) \end{aligned}$$

by the positivity of the Hilbert Space. The energy of any non-vacuum state is positive-definite, and the vanishing of the vacuum state is a necessary condition for the existence of a unique vacuum.

2.2.2 Equal Number of Bosons and Fermions

Defining a fermionic number operator $\nu_F = (-1)^{N_F}$, where

$$\begin{aligned} N_F \left| B \right\rangle &= (\text{even number}) \left| B \right\rangle \\ N_F \left| F \right\rangle &= (\text{odd number}) \left| F \right\rangle , \end{aligned}$$

gives

$$\nu_F = \begin{cases} +1 & \text{bosonic state} \\ -1 & \text{fermionic state} \end{cases}$$

This operator commutes with the supercharges. Using the SUSY identity and taking the trace

$$\begin{aligned} Tr(\nu_F \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}) &= Tr(\nu_F Q_\alpha \bar{Q}_{\dot{\alpha}} + \nu_F \bar{Q}_{\dot{\alpha}} Q_\alpha) \\ &= Tr(-Q_\alpha \nu_F \bar{Q}_{\dot{\alpha}} + Q_\alpha \nu_F \bar{Q}_{\dot{\alpha}}) \\ &= 0. \end{aligned}$$

where we've made use of the cycle property of the trace. Looking at the RHS of $\nu_F\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\nu_F \sigma^\mu_{\alpha\dot{\alpha}} P_\mu$, this implies that $Tr\nu_F = 0$. The trace of the fermionic operator can be expressed as

$$\begin{aligned} Tr\nu_F &= \sum_{bosons} \left\langle B \right| \nu_F \left| B \right\rangle + \sum_{fermions} \left\langle F \right| \nu_F \left| F \right\rangle \\ &= \sum_{bosons} \left\langle B \right| \left| B \right\rangle - \sum_{fermions} \left\langle F \right| \left| F \right\rangle . \\ &= 0 \end{aligned}$$

Thus, for a given representation, known as a multiplet, the number of bosons and fermions will be the same; we must have the same degrees of freedom for either within the same multiplet.

$$\sum_{bosons} \left\langle B \right| \left| B \right\rangle = \sum_{fermions} \left\langle F \right| \left| F \right\rangle . \tag{5}$$

2.2.3 Equal Masses

The mass operator $P_\mu P^\mu = P^2 = -\mathcal{M}^2$ is a Casimir of the super-Poincaré group, and the mass of particles in any given representation of SUSY is the same[30]. As a Casimir, P^2 commutes with all other generators of the algebra. We consider the supercharge Q_α, which transforms bosons into fermions and vise versa, i.e. for bosonic state $\left| B \right\rangle$ and fermionic state $\left| F \right\rangle$

$$Q_\alpha \left| B \right\rangle = \left| F \right\rangle .$$

We have $\left[P^2, Q_\alpha\right] = 0$, giving

$$\begin{aligned} [P^2, Q_\alpha] \left| B \right\rangle &= 0 \\ P^2 Q_\alpha \left| B \right\rangle - Q_\alpha P^2 \left| B \right\rangle &= 0 \\ P^2 \left| F \right\rangle - m_B^2 \left| B \right\rangle &= 0 \\ m_F^2 \left| F \right\rangle - m_B^2 \left| B \right\rangle &= 0 \end{aligned}$$

.
SUSY thus says that the mass of the fermions must be equal to the mass of the bosons in a given representation

$$m_F^2 = m_B^2. \tag{6}$$

Of course, this symmetry isn't realised in nature, so must be spontaneously broken at low enough energies. One may also ask about the spin of the particles, do we have the same spin in a given representation? The Poincaré group has $W^2 = W^\mu W_\mu$ as a Casimir, where W_μ is the Pauli-Lubanski vector that describes spin, and thus by Wigner's classification it is possible to label a representation by its spin. However, it is not a Casimir of the super-Poincaré group[5], and thus different particles within the same representation will have different spins.

2.3 Representations of Supersymmetry

Using the analysis and consequences from the previous section, the representations of SUSY may be built. In the Poincaré group, we classify particles by their properties. The super-Poincaré group is a larger group, but still contains the Poincaré group as a subgroup. Using Wigner's classification, we can define a superparticle, which will be a collection of particles, known as a *multiplet*[5]. Initially, the simplest case of $\mathcal{N} = 1$ is focused upon, before making the natural progression to $\mathcal{N} \geq 1$, known as extended SUSY.

2.3.1 $\mathcal{N} = 1$ Supersymmetry

Wigner's Classification may be used to find the unitary irreducible representations (irreps) of SUSY, adopting the usual method of moving into the rest frame by setting $P^2 = 0$. For the purposes of this paper, we focus on the massless representations, with rest frame momentum $P^\mu = (E, 0, 0, E)$. Once again, using equation (1) gives

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2E \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right]_{\alpha\dot{\alpha}} = 4E \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\alpha}}.$$

From the positivity of energy, we see that half of the operators are trivially realised. The remaining half can be used to suggestively define raising and lowering operators

$$a = \frac{1}{\sqrt{4E}} Q_\alpha \quad a^\dagger = \frac{1}{\sqrt{4E}} \bar{Q}_{\dot{\alpha}},$$

analogous to those found in quantum theory, where $\{a, a^\dagger\} = \mathbb{I}$. Different particles within the same representation will have different spin (or helicity for the massless case), and we can use these eigenvalues as a basis to create the multiplet of particles. The spin generators are defined by $J_i = \epsilon_{ijk} M_{jk}$. Choosing arbitrary $J_3 = M_{12}$ and using the SUSY algebra

$$\begin{aligned} [M_{12}, Q_2] &= i(\sigma_{12})_2^2 Q_2 \\ &= \frac{1}{2}(\sigma_3)_2^2 Q_2 \\ &= -\frac{1}{2} Q_2, \end{aligned}$$

indicates that the supercharge lowers the helicity by half. Here we used $\sigma_{12} = \frac{1}{4}(\sigma_1\bar{\sigma}_2 - \sigma_2\bar{\sigma}_1) = -\frac{i}{2}\sigma_3$. Similarly, $\bar{Q}_{\dot{\alpha}}$ raises the helicity by half; these are the building blocks of the representations. We define a vacuum state $|\lambda_0\rangle$, known as the Clifford vacuum, such that it is annihilated by the supercharge Q_α

$$Q_\alpha |\lambda_0\rangle = 0. \tag{7}$$

The eigenvalue of this state, λ_0, is the state of minimal helicity, upon which we build our representations. For $\mathcal{N} = 1$ SUSY we only have one copy of the supercharges, and thus may only have two states in the representation

$$\begin{aligned} & |\lambda_0\rangle \\ \bar{Q}_{\dot{\alpha}} |\lambda_0\rangle \equiv & |\lambda_0 + \frac{1}{2}\rangle , \end{aligned}$$

differing by $\frac{1}{2}$ in helicity. Dependent on the choice of minimal helicity, we can have various multiplets, including:

- $\lambda_0 = 0$: within this multiplet lies a scalar and fermion of helicity $+\frac{1}{2}$, and is known as the *chiral multiplet.*
- $\lambda_0 = \frac{1}{2}$: within this multiplet likes a fermion and vector field, and is known as the *vector multiplet.* Particles within the same representation must transform similarly, and as this multiplet contains the vector field, it transforms in the adjoint representation.
- $\lambda_0 = \frac{3}{2}$: within this multiplet we have a gravitino and graviton, and is known as the *gravity multiplet.*

We're restricted to a maximum helicity of 2, as there is no known consistent local interacting theory that contains spins greater than this. Finally, we must consider the CPT invariance of the multiplet. Under this symmetry, the helicity changes sign, and therefore it would be broken without a symmetric distribution of helicity around 0[5]. We know that this symmetry isn't broken in nature, and must include the CPT conjugate multiplet to give

$$\begin{aligned} \left(0, \frac{1}{2}\right) & \oplus \left(-\frac{1}{2}, 0\right) \\ \left(\frac{1}{2}, 1\right) & \oplus \left(-1, \frac{1}{2}\right) \\ \left(\frac{3}{2}, 2\right) & \oplus \left(-2, -\frac{3}{2}\right) \end{aligned} \tag{8}$$

2.3.2 Extended Supersymmetry

We now turn our attentions to extended $\mathcal{N}$ SUSY, where we have more than one copy of a supercharge Q^I_α, $I = 1, \ldots, \mathcal{N}$. Using the previous method, we can define the Clifford vacuum $|\lambda_0\rangle$, and build representations using the annihilation and creation operators

$$\begin{aligned} a^I &= \frac{1}{\sqrt{4E}} Q^I_\alpha \\ a^{I\dagger} &= \frac{1}{\sqrt{4E}} \bar{Q}^I_{\dot{\alpha}}. \end{aligned}$$

Because there are $\mathcal{N}$ copies, we can build upon the Clifford vacuum in various ways

$$
\begin{aligned}
a^{1\dagger}|\lambda_0\rangle &= |\lambda_0+\frac{1}{2}\rangle^1 \\
a^{2\dagger}a^{1\dagger}|\lambda_0\rangle &= |\lambda_0+1\rangle^{12} \\
&\dots \\
a^{N\dagger}\dots a^{1\dagger}|\lambda_0\rangle &= |\lambda_0+\frac{\mathcal{N}}{2}\rangle^{1\dots\mathcal{N}} \quad .
\end{aligned}
$$

The operators are antisymmetric in these indices, and for a given helicity $\lambda_0+\frac{k}{2}$ there are $\binom{\mathcal{N}}{k}$ options[4]. The total number of states is therefore given by the binomial formula

$$(x+y)^{\mathcal{N}}=\sum_{k=0}^{\mathcal{N}}\binom{\mathcal{N}}{k}x^{\mathcal{N}-k}y^k,$$

applying this to our case

$$2^{\mathcal{N}}=(1+1)^{\mathcal{N}}=\sum_{k=0}^{\mathcal{N}}\binom{\mathcal{N}}{k}1^{\mathcal{N}-k}1^k=\sum_{k=0}^{\mathcal{N}}\binom{\mathcal{N}}{k}, \tag{9}$$

gives a total of $2^{\mathcal{N}}$ states in a representation, and by the consequence of SUSY, $2^{\mathcal{N}-1}$ are bosons and the other $2^{\mathcal{N}-1}$ are fermions. What multiplets do we have for $\mathcal{N}>1$ SUSY?

2.3.3 $\mathcal{N}=2$ Representations

We have four possible representations:

- $\lambda_0=-\frac{1}{2}\rightarrow\left(-\frac{1}{2},0,0,\frac{1}{2}\right)\oplus\left(-\frac{1}{2},0,0,\frac{1}{2}\right)$: the *hypermultiplet,* containing two Weyl fermions and two complex scalars
- $\lambda_0=0\rightarrow\left(0,\frac{1}{2},\frac{1}{2},1\right)\oplus\left(-1,-\frac{1}{2},-\frac{1}{2},0\right)$: the *gauge* or *vector multiplet*, containing one vector, two Weyl fermions, and a complex scalar
- $\lambda_0=\frac{1}{2}\rightarrow\left(\frac{1}{2},1,1,\frac{3}{2}\right)\oplus\left(-\frac{3}{2},-1,-1,-\frac{1}{2}\right)$: the *gravitino multiplet,* containing one gravitino, two vectors, and a Weyl fermion.
- $\lambda_0=1\rightarrow\left(1,\frac{3}{2},\frac{3}{2},2\right)\oplus\left(-2,-\frac{3}{2},-\frac{3}{2},-1\right)$: the *graviton multiplet,* containing one graviton, two gravitini and a vector.

2.3.4 $\mathcal{N}=4$ Representations

Here we have only one possible representation, the $\mathcal{N}=4$ *gauge multiplet,* which contains the following

λ	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
#	1	4	6	4	1

(10)

This multiplet is known as the $\mathcal{N}=4$ Super Yang-Mills multiplet[8].

2.4 $\mathcal{N}=4$ Super Yang-Mills Theory

$\mathcal{N}=4$ Super Yang-Mills Theory is a non-Abelian theory - as all fields are within the same multiplet, they cannot escape the gauge field, and must all transform within the adjoint representation, with gauge group $SU(N)$.

$$\mathcal{L} = tr\Big\{ \frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a \bar{\sigma}^i D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i + \sum_{a,b,i} gC_i^{ab}\lambda_a \left[X^i, \lambda_b\right] + \sum_{abi} g\bar{C}_{iab}\bar{\lambda}^a \left[X^i, \bar{\lambda}_b\right] + \sum_{abi} g\bar{C}_{iab}\bar{\lambda}^a \left[X^i, \bar{\lambda}^b\right] + \frac{g^2}{2}\sum_{ij}\left[X^i, X^j\right]^2 \Big\}, \tag{11}$$

where D_μ is just the usual gauge covariant derivative, of the form $D_\mu = \partial_\mu + i\left[A_\mu, \right]$. Here, $\tilde{F}_{\mu\nu}$ is the hodge dual of the field strength $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, and appears in the Lagrangian as an instanton term, controlled by the instanton angle θ_I [3], which under $\theta_I \to \theta_I + 2\pi$ remains invariant. The number of degrees of freedom in the table denotes the transformation properties of the fields under the $SU(4)_R$ symmetry: the vector A_μ transforms in the singlet representation $\underline{\mathbf{1}}$, the fermions λ_α in the (anti)fundamental representations $\underline{\mathbf{4}}$,$\underline{\bar{\mathbf{4}}}$, and the scalars X_i in the adjoint representation $\underline{\mathbf{6}}$. The algebra of $SU(4)$ is given by the Dynkin Diagram A_3

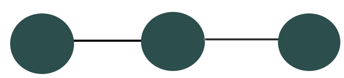

Figure 1: Dynkin diagram A_3 representing the Lie algebra of $SU(4)$

whose representations can be classified in terms of Dynkin labels $[n_1, n_2, n_3]$. The dimensions of the representations are given by Weyl's dimension formula:

$$dim[n_1, n_2, n_3] = \frac{1}{12}(n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(n_2+n_3+2)(n_1+n_2+n_3+3) \tag{12}$$

- The scalars are represented by [010]
- The fermions are represented by [001]

The transformations under SUSY are given as

$$\begin{aligned}
\delta X^i &= \left[Q^a_\alpha, X^i\right] = C^{iab}\lambda_{\alpha b} \\
\delta \lambda_b &= \left\{Q^a_\alpha, \lambda_{\beta b}\right\} = F^+_{\mu\nu}(\sigma^{\mu\nu})^\alpha{}_\beta \delta^a_b + \left[X^i, X^j\right]\epsilon_{\alpha\beta}(C_{ij})^a{}_b \\
\delta \bar{\lambda}^b_{\dot{\beta}} &= \left\{Q^a_\alpha, \bar{\lambda}^b_{\dot{\beta}}\right\} = C_i^{ab}\bar{\sigma}^\mu_{\alpha\dot{\beta}} D_\mu X^i \\
\delta A_\mu &= \left[Q^a_\alpha, A_\mu\right] = (\sigma_\mu)_\alpha{}^{\dot{\beta}}\bar{\lambda}^a_{\dot{\beta}}.
\end{aligned}$$

where the constants $(C_{ij})^a{}_b$ are related to the Clifford Dirac matrices of $SO(6)_R$[8]. It's clear to see that, due to the action of the supercharge, a boson will always transform to a fermion, and vise versa. Because the action must be dimensionless, we can deduce the dimensions of the fields and the couplings (in terms of mass dimension).

$$
\begin{aligned}
S &= \int d^4x\mathcal{L} \\
\left[\int d^4x\right] &= -4 \implies [\mathcal{L}] = 4.
\end{aligned}
$$

From this we see deduce that $[X] = 1$ and the coupling constant g is dimensionless. Thus there is no running of the coupling, and our Super Yang-Mills theory is scale invariant, which is quite unique amongst field theories. The relationship between scale and conformal invariance is a difficult one; whilst there is no proof that they're equivalent, it is believed that if a theory demonstrates scale invariance, it will also be invariant under the conformal group, becoming a conformal field theory[21]. This is the case for $\mathcal{N} = 4$ Super Yang-Mills theory.

3 Conformal and Superconformal Symmetry Group

3.1 The Conformal Group

The conformal group in Euclidean space is one whose transformations preserve the angles of the space. We can extend this to Minkowski spacetime, where the conformal group preserves the form of the metric up to a scale factor, thus preserving the causal structure of the spacetime. We define the conformal transformation map from a spacetime (M, g) to $(M, \tilde{g})$, where M is the manifold and g is the metric, such that the metric transforms as $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Lambda^2(x)g_{\mu\nu}$, where $\Lambda^2(x)$ is a smooth function that is positive everywhere.
The conformal group is a bosonic extension of the Poincaré group, and is generated by[1]:

- Lorentz transformations $M_{\mu\nu}$
- Translations P_μ
- Scaling D
- Special Conformal transformations K_μ

As we did for the super-Poincaré group, we can study the structure of the algebra by looking at the infinitesimal transformations, beginning with scalings and translations. We consider the product of two transformations - first translations and then scaling, and vise versa. The transformations are:

1. $x_\mu \to x'_\mu = (x_\mu + \epsilon_\mu)$; $x'_\mu \to x''_\mu = (1+\lambda)(x_\mu + \epsilon_\mu) = x_\mu + \lambda x_\mu + \epsilon_\mu + \lambda\epsilon_\mu$
2. $x_\mu \to x'_\mu = x_\mu + \lambda x_\mu$; $x'_\mu \to x''_\mu = x_\mu + \lambda x_\mu + \epsilon_\mu$,

which can be infinitesimally expressed as

1. $U(\lambda T) = \mathbb{I} + i\lambda D - i\eta^{\nu\mu}(\epsilon_\nu + \lambda\epsilon_\nu)P_\mu$
2. $U(T\lambda) = \mathbb{I} + i\lambda D - i\eta^{\nu\mu}\epsilon_\nu P_\mu$.

Each transformation has an associated unitary operator; scalings given by $U(\lambda) = e^{i\lambda D}$ and translations by the familiar $U(T) = e^{-i\epsilon^\mu P_\mu}$. Treating these infinitesimally also, we find

$$
\begin{aligned}
U(\lambda)U(T) - U(T)U(\lambda) &= \eta^{\nu\mu}\lambda\epsilon_\nu[D, P_\mu] \\
U(\lambda T) - U(T\lambda) &= -i\eta^{\nu\mu}\lambda\epsilon_\nu P_\mu.
\end{aligned}
$$

Equating the RHS of each equation gives

$$[D, P_\mu] = -iP_\mu, \tag{13}$$

and with similar analysis we also find

$$[D, K_\mu] = iK_\mu. \tag{14}$$

The interesting representations to focus on are those build around the eigenfunctions of the scaling operator D, with eigenvalue $-i\Delta$, where Δ is the scaling dimension[1]. In this representatuion, P_μ raises the state and K_μ lowers it. Wtih this structure, we define the *primary operator* to be the state which is annihilated by K_μ, and upon which all other states in the representation may be built upon. The full structure of the algebra is given by

$$[M_{\mu\nu}, P_\mu] = -i(\eta_{\mu\nu}P_\nu - \eta_{\nu\rho}P_\mu) \qquad [M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho})$$

$$[M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \qquad [P_\mu, K_\mu] = 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D.$$

The algebra may be packaged in a suggestive way by defining new generators[1]

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu), \quad J_{\mu(d+1)} = \frac{1}{2}(K_\mu + P_\mu), \quad J_{(d+1)d} = D,$$

and defining the matrix

$$J_{ab} = \begin{pmatrix} M_{\mu\nu} & \frac{1}{2}(K_\mu - P_\mu) & \frac{1}{2}(K_\mu + P_\mu) \\ -\frac{1}{2}(K_\mu - P_\mu) & 0 & D \\ -\frac{1}{2}(K_\mu + P_\mu) & D & 0 \end{pmatrix} \tag{15}$$

The matrix may look familiar - it is a $(d{+}2)\times(d{+}2)$ antisymmetric matrix with the Lorentz subgroup. We see that the conformal group is isomorphic to $SO(2, d)$; or in four dimensions $SO(2, 4)$. This is a key point, and should be noted - we will see that this is also the isometry group of AdS_5. It teases us with a possible relation between a $(d{+}1)$-dimensional gravity theory in AdS space and a d-dimensional conformal field theory. The full beauty will be revealed in the following sections.

3.2 The Superconformal Group

We have seen both a fermionic and bosonic extension of the Poincaré group, the former in the guise of the super-Poincaré group and the latter in the Conformal group. The natural question from here is to ask whether there is a way to merge the extensions into one encompassing algebra (as I mentioned, phyiscists always like lesser parts!). This is possible, and is called the Superconformal Group. The algebra inherits the generators from both the SUSY algebra and Conformal group, and we find two additional generators that the algebra closes on: another fermionic generator S_α, and its hermitian conjugate, and R-symmetry, which is no longer an automorphism of the group[8, 1]. As this is incorporated into the algebra, we must also absorb the symmetry group, $SU(4) \sim SO(6)$, consequently giving the superconformal group as $SO(4, 2) \times SU(4) \equiv SU(2, 2 \mid 4)$, with $SO(4, 2) \sim SU(2, 2)$.
The structure of the algebra can be given schematically as[1]

$$[K, \bar{Q}] \sim S \qquad [K, Q] \sim \bar{S} \qquad [D, Q] \sim -\frac{i}{2}Q$$

$$[D, S] \sim \frac{i}{2}S \qquad [P, S] \sim Q \qquad \{S, S\} \sim K \qquad \{Q, S\} \simeq M + D + R$$

.

Once again, the interesting representations are those based around the eigenfunctions of the scaling generator D. Analogous to the conformal group, the state that is annihilated by S_α is defined to be the *superprimary operator,* which is the state with lowest scaling dimension. We build the remaining states of the representation by repeated action with the supercharges Q_α, which always raises the scaling dimension, and so it is not possible to write the superprimary operator in terms of a commutator with Q_α. So what options do we have for a superprimary operator in the Super Yang-Mills theory? Looking back at the SUSY transformation of the fields, we see that we cannot use terms including the derivatives of X^i, fermions, or commutators of X^i, as all these can be expressed in terms of a commutator of Q_α. What we are left with is the symmetric part of the X^i, and consider the simplest form, known as a single trace operator

$$O^{i_1 \dots i_n} = sTr(X^{i_1} \dots X^{i_n}), \tag{16}$$

where sTR is the symmetrised trace, and the index $i_j, j = 1 \dots N$ runs over the $SO(6)_R$ fundamental representation that the scalars transform in.
We may also have some superprimary operators that are annihilated by some of the supercharges, and these are known as *chiral superprimary operators*, and they form the basis of a chiral, or 'BPS' multiplet. With fewer supercharges, we have fewer building blocks for our representation, and therefore the multiplets become shorter. These multiplets are known as *chiral* multiplets. To determine whether we have a chiral multiplet we look at the superprimary operator and its tranformation under the R-symmetry group. The scalars are in the $\underline{6}$ of $SU(4)_R$, characterised by the representation [010]. We can consider the symmetrised product of two scalars $Sym^2[010] = [020] + [000]$ or, in terms of dimensions, $\frac{(6.7)}{2} = \underline{\mathbf{21}} \to \underline{\mathbf{20}} + \underline{\mathbf{1}}$. The $\underline{\mathbf{1}}$ rep corresponds to $sTr(X^i X^i)$, and to check for null vectors, we look at the commutation relations between this and the supercharges, as for chiral multiplets, we know the superprimary operator is annihilated by some of the supercharges. Schematically, we find $[Q_\alpha, Tr(X^i X^i)] \sim Tr(\lambda_\alpha X^j)$, $\underline{\mathbf{4}} \times \underline{\mathbf{1}} \to \underline{\mathbf{4}}$, so the representation sizes tie out on either side. However, if we look at the $\underline{\mathbf{20}}$ rep, which corresponds to the symmetric traceless product $sTr(X^{\{i} X^{j\}}$, we find $[Q_\alpha, sTr(X^{\{i} X^{j\}}] \sim Tr(\lambda_\alpha X^j)$, where the LHS should give $\underline{\mathbf{4}} \times \underline{\mathbf{20}} \to \underline{\mathbf{60}} + \underline{\mathbf{20}}$, but we only see the $\underline{\mathbf{20}}$ on the RHS[1]. Therefore we see that we have null vectors, corresponding to a short multiplet. It turns out that the symmetric traceless product of scalars will always be the superprimary of a chiral multiplet.

3.3 Properties of Correlation Functions

The aim of the correspondence is to relate the observables of both the gravity and conformal field theories, for which the latter includes correlation functions. In section 7, the gravity theory dynamics of the bulk field in AdS_5 can be shown to determine the two- and three-point correlation functions in the Super Yang-Mills theory, which is invariant under the conformal group. As this group has a high degree of symmetry, including

- Translational invariance
- Scaling
- Special Conformal,

it imposes restrictions on the forms of the correlations functions. Infact, for two- and three-point functions, the symmetry completely determines the structure. It is important to ensure that the correspondence produces the results that obey the conformal group.

3.3.1 Two-Point Functions

The two-point functions may be represented by $\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = g(x_1, x_2)$. The forms imposed by the symmetries are:

- Translational invariance: as this is an invariant under $f(x) = x + a$, g must be a function of the difference between two vectors

$$g(x_1, x_2) = g(x_1 - x_2).$$

- Scaling: consider two operators of conformal dimension Δ_1, Δ_2. The two-point function must be invariant under rescalings $f(x) = \lambda x$:

$$\langle \lambda^{\Delta_1} \mathcal{O}(\lambda x_1) \lambda^{\Delta_2} \mathcal{O}(\lambda x_2) \rangle = \lambda^{\Delta_1 + \Delta_2} g(\lambda(x_1 - x_2)) = g(x_1 - x_2).$$

due to the previous restriction by translational invariance[13]. This forces us to choose the form

$$g(x_1 - x_2) = \frac{d_{12}}{(x_1 - x_2)^{\Delta_1 + \Delta_2}},$$

where d_{12} is known as the structure constant.

- Special Conformal Transformations: these are a combination of translations and inversions. We have already restricted the form of the two-point function with translational invariance, so just need to consider the restrictions imposed by inversions $f(x) = -\frac{1}{x}$.

$$\langle \frac{1}{x_1^{2\Delta_1}} \frac{1}{x_2^{2\Delta_2}} \mathcal{O}(-\frac{1}{x_1}) \mathcal{O}(-\frac{1}{x_2}) \rangle = \frac{1}{x_1^{2\Delta_1} x_2^{2\Delta_2}} \frac{d_{12}}{\left(-\frac{1}{x_1} + \frac{1}{x_2}\right)^{\Delta_1 + \Delta_2}} = \frac{d_{12}}{(x_1 - x_2)^{\Delta_1 + \Delta_2}}.$$

To obey the restrictions of scaling, the conformal weights of the two operators must be the same, $\Delta_1 = \Delta_2 \equiv \Delta$, giving

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{d_{12}}{(x_1 - x_2)^{2\Delta}}. \tag{17}$$

This is the expected form of the two-point function for an operator of conformal weight Δ.

3.3.2 Three-Point Functions

The three point functions may be represented by $\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \rangle = g(x_1, x_2, x_3)$. The forms imposed by the symmetries are

- Translational Invariance:

$$g(x_1, x_2, x_3) = g((x_1 - x_2)(x_2 - x_3)(x_1 - x_3)).$$

- Scaling invariance:

$$\langle \lambda^{\Delta_1} \mathcal{O}(\lambda x_1 \lambda^{\Delta_2} \mathcal{O}(\lambda x_2) \lambda^{\Delta_3} \mathcal{O}(\lambda x_3) \rangle = \lambda^{\Delta_1 + \Delta_2 + \Delta_3} g(\lambda(x_1 - x_2)(x_2 - x_3)(x_1 - x_2)) = g((x_1 - x_2)(x_2 - x_3)(x_1 - x_3)),$$

which tells us that the three-point function must be of the form

$$\frac{C_{123}}{(x_1 - x_2)^a (x_2 - x_3)^b (x_1 - x_3)^c},$$

where C_{123} is the structure constant, and $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$ to cancel out the λ in the numerator.

- Finally the special conformal transformations impose our choices of a, b and c and we find

$$
\begin{aligned}
a &= \Delta_1 + \Delta_2 - \Delta_3 \\
b &= \Delta_2 + \Delta_3 - \Delta_1 \\
c &= \Delta_1 + \Delta_3 - \Delta_2,
\end{aligned}
$$

which gives the conformal structure of the three-point function as

$$
\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = \frac{C_{123}}{|\ \overrightarrow{x_1} - \overrightarrow{x_2}\ |^{\Delta_1+\Delta_2-\Delta_3}|\ \overrightarrow{x_1} - \overrightarrow{x_3}\ |^{\Delta_1+\Delta_3-\Delta_2}|\ \overrightarrow{x_2} - \overrightarrow{x_3}\ |^{\Delta_2+\Delta_3-\Delta_1}}. \tag{18}
$$

We will see in section 7 that the forms of the correlation functions from the gravity theory exactly match what we would expect for correlation functions these conformal dimensions.

3.4 't Hooft Limit and Large N

The origins of string theory lie far away from where they are now. The theory was originally formulated to describe the strong force, but the presence of a massless spin-two particle and the demanding requirement that we live in ten dimensions meant it was abandoned in favour of QCD, which has held up very well ever since. In his 1974 paper, 't Hooft showed insight into a link between strings and QCD that brought the theory back to its roots. It is an aymptotically free theory, so whilst it is easy to describe the high-energy behaviour of the strong interactions with a gauge theory, this becomes more difficult at lower energies due to the increasing strength of the coupling constant, meaning that pertubative expansions can't be used. However, 't Hooft looked to expanding the $SU(3)$ colour group to arbitary N colours with $SU(N)$ gauge group, looking at the topological properties of the Feynman diagrams. By defining a fixed coupling constant $\lambda = g^2 N$, the 't Hooft coupling, whilst this limit is taken, we can then define a perturbative expansion in $1/N$[27].
We may think of some generic non-Abelian theory for some field Φ_i^a, where a is in the adjoint representation of $SU(N)$, and i denotes some particular field (i.e. a scalar, quark, gauge). We assume the Lagrangian includes three-point and four-point interactions, and by rescaling the fields $\tilde{\Phi} = g\Phi$ can write a schematic Lagrangian as

$$
\mathcal{L} \sim \frac{1}{g^2}\left(Tr(\tilde{\Phi}_i\tilde{\Phi}_i) + Tr(\tilde{\Phi}_i\tilde{\Phi}_j\tilde{\Phi}_k) + Tr(\tilde{\Phi}_i\tilde{\Phi}_j\tilde{\Phi}_k\tilde{\Phi}_l)\right) \tag{19}
$$

To acquire the Feynman diagrams for this theory, double-line notation is used. The fields in the adjoint representation are considered as the product between the fundamental and anti-fundamental representations, and thus we label the field as Φ_j^i, with one index in each representation, and draw this on the diagram as two double lines in opposing directions. Considering the Feynman diagrams as a topological simplical triangulation, we have the contributions[18]:

- propagators contribute $\frac{\lambda}{N}$, which we denote E, the edges of double lines
- vertices contribute $\frac{N}{\lambda}$, denoted V
- loops contribute N, denoted F, as the loops form faces

The contribution to each diagram is given as $N^{F+V-E}\lambda^{E-V}$. However, in terms of topological expansions, the quantity $F + V - E$ is equal to Euler's characteristic, χ which for closed oriented surfaces can be written as $\chi = 2 - 2g$, g being the genus of the surface[20].
Doing a perturbative expansion over the diagrams gives

$$
\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda), \tag{20}
$$

where $f_g(\lambda)$ is some polynomial. As we take the large N limit, $N \to \infty$, the graphs with the lowest genus dominate, indicating that the theories for large N should be simpler than, say, $SU(3)$. This is the same perturbative expansion as in closed oriented string theory, if we make the idenficiation $g_s = \lambda/N$, where g_s is the string coupling. We see a mapping between the gauge theory and string theory, adding another tantalising element to Malcadena's conjecture.

4 Anti de-Sitter space

Anti de-Sitter space, known as AdS, is a Lorentzian analogue of Euclidean hyperbolic space, holding constant negative curvature. The correspondence uses the idea of the holographic principle, which states that some $(d+1)$-dimensional theory can be encoded entirely at the d-dimensional boundary. To make the connection between the gravity and conformal field theories, we must understand the way AdS behaves both in the bulk and at the boundary, where the conformal field theory lives. This chapter looks at the conformally compactified structure of both flat space and AdS, in order to make the necessary geometric relation between the two. Within this chapter we follow the method of [9]

4.1 Conformal Compactification

The method of conformal compactification is useful for the investigation of the causal structure of spacetime. In this, infinite distances are brought to a finite coordinate distance, allowing us to probe the structure at the boundaries of the spacetime. Integral to this method is the idea of a conformal transformation, which takes a spacetime equipped with a metric g to another, equipped with a metric $\tilde{g}$, related to the original metric by

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \tag{21}$$

where $\Omega(x)$ is known as the conformal factor, and is taken as $\Omega^2(x) > 0$ everywhere within the original coordinate series. In order to bring the diverging distances in the metric to within a finite distance, the factor must simultaneously decrease, eventually reaching zero on the conformal boundary only defined for $\tilde{g}_{\mu\nu}$. Conformal transformations preserve the causal structure of spacetime, such that vectors which were defined to be timelike/null/spacelike with the original metric, continue to be so with the conformally transformed metric. As mentioned in the previous section, one of the points at the heart of the correspondence is the equivalence between the isometry group of AdS_{d+1} and the conformal symmetry of flat Minkowski space $\mathbb{R}^{1,d-1}$. To see this, we begin by looking at the conformal structure of Minkowski space.

4.1.1 Conformal Structure of 2D Minkowski Space, $\mathbb{R}^{1,1.}$

The metric for 2D Minkowski space is

$$ds^2 = -dt^2 + dx^2, \tag{22}$$

with the coordinates defined over the range $-\infty < t, x < \infty$. With these coordinates ranges, there is no way to represent the spacetime on a finite piece of paper, the infinite points are not defined within the spacetime - to do this we must compactify. The first step is to transform into Eddington-Finkelstein coordinates, based on null geodesics (lightcone coordinates)

$$\begin{aligned} t + x &\equiv v \quad \text{ingoing EF coordinate} \\ t - x &\equiv u \quad \text{outgoing EF coordinate.} \end{aligned}$$

With this coordinate transformation, the metric reads

$$ds^2 = -dudv.$$

We can then transform into trigonometric coordinates $u = tan\tilde{u}$ and $v = tan\tilde{v}$, such that

$$-\infty < u, v < \infty \quad \leftrightarrow \quad -\frac{\pi}{2} < \tilde{u}, \tilde{v} < \frac{\pi}{2}$$

$$\frac{du}{d\tilde{u}} = \frac{d}{d\tilde{u}}(tan\tilde{u}) = \frac{d}{d\tilde{u}}(\frac{sin\tilde{u}}{cos\tilde{u}}) = \frac{cos^2\tilde{u} + sin^2\tilde{u}}{cos^2\tilde{u}} = \frac{1}{cos^2\tilde{u}},$$

giving

$$ds^2 = -\frac{1}{cos^2\tilde{u}cos^2\tilde{v}}d\tilde{u}d\tilde{v}.$$

Making the conformal transformation $\tilde{g}_{\mu\nu} = (cos\tilde{u}cos\tilde{v})^2 g_{\mu\nu}$, we define a new metric $\tilde{ds}^2 = -d\tilde{u}d\tilde{v}$. The range has been brought to within a finite coordinate distance with the use of trigonometric coordinates. The conformally transformed metric is regular at $\pm\frac{\pi}{2}$, and can be added as points to create a new spacetime, denoted $\tilde{M}$. The original Minkowski spacetime is a subset of this spacetime, which we define as the conformal compactification of M.

4.1.2 Conformal Structure of Higher-Dimensional Minkowski Space, $\mathbb{R}^{1,d-1.}$

The previous example can be extended to higher dimensional Minkowski space, with metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2_{d-2}.$$

$d\Omega^2_{d-2}$ is the spatial metric defined on the $d-2$ sphere. We follow the same logic as before, but with an additional constraint - we have introduced a radial coordinate, r, which is defined only for $r \geq 0$. Consequently, we find that $v \geq u$, and thus $\tilde{v} \geq \tilde{u}$. The conformally transformed metric now reads

$$\tilde{ds}^2 = -4\tilde{du}\tilde{dv} + sin^2(\tilde{v} - \tilde{u})d\Omega^2_{d-2}.$$

However, this is not the most useful form, and to make the conformal structure upon compactification manifest, we make the transformation $\tau = \tilde{v} + \tilde{u}$ and $\chi = \tilde{v} - \tilde{u}$.

$$\begin{aligned}
d\tau &= d\tilde{v} + d\tilde{u} \\
d\chi &= d\tilde{v} - d\tilde{u} \\
-d\tau^2 &= (d\tilde{v} + d\tilde{u})^2 \\
d\chi^2 &= (d\tilde{v} - d\tilde{u})^2 \\
-d\tau^2 + d\chi^2 &= -4d\tilde{u}d\tilde{v} \\
ds^2 &= -d\tau^2 + d\chi^2 + sin^2\chi d\Omega^2_{d-3}.
\end{aligned}$$

This gives the topology of $\mathbb{R} \times S^{d-2}$, which is the same geometry as Einstein's Static Universe[29]. This will be important when we look at the conformal compactification of AdS_{d+1} in the next subsection.

4.1.3 Conformal Structure of Anti-de Sitter Space

AdS_d can be represented by the hyperboloid embedded in $d+1$ dimensional space[22]

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} = R^2$$

$$ds^2 = -dX_0^2 - dX_d^2 + \sum_{i=1}^{d-1} dX_i^2,$$

where we have defined some radius of curvature, R. Orthogonal transformations preserve the magnitude of a vector, and the embedding of AdS_d gives the same equation as the invariant of a rotation in

$d+1$ dimensional space, making the $SO(2, d-1)$ symmetry manifest. The coordinates that satisfy the above are

$$\begin{aligned} X_0 &= Rcosh\rho cos\tau \\ X_d &= Rcosh\rho sin\tau \\ X_i &= Rsinh\rho\Omega_i, \end{aligned}$$

where we constrain Ω_i by imposing $\sum_i \Omega_i^2 = 1$.With this choice of coordinates, we find that the differentials are

$$\begin{aligned} dX_0(\rho,\tau) &= Rsinh\rho cos\tau d\rho - Rcosh\rho sin\tau d\tau \\ dX_d(\rho,\tau) &= Rsinh\rho sin\tau d\rho - Rcosh\rho cos\tau d\tau \\ dX_i(\rho,\Omega_i) &= Rcosh\rho\Omega_i d\rho + Rsinh\rho d\Omega_i. \end{aligned}$$

So, our metric becomes (by noting $\Omega_i d\Omega_i = 0$)

$$ds^2 = R^2(-cosh^2\rho d\tau^2 + d\rho^2 + sinh^2\rho d\Omega_{d-2}^2). \tag{23}$$

These coordinates are called global coordinates, and cover the whole space, with $\rho \geq 0$ and $0 \leq \tau < 2\pi$. This is quite a suggestive coordinate system, and by making the transformation $tan^2\theta = sinh^2\rho$ we see

$$\begin{aligned} cosh^2\rho - sinh^2\rho &= 1 \\ \frac{sin^2\theta + cos^2\theta}{cos^2\theta} &= tan^2\theta + 1 \\ tan^2\theta + 1 &= sinh^2\rho + 1 \\ \frac{1}{cos^2\theta} &= cosh^2\rho, \end{aligned}$$

which gives the metric

$$ds^2 = \frac{R^2}{cos^2\theta}(-d\tau^2 + d\theta^2 + sin^2\theta d\Omega_{d-2}^2).$$

The prefactor can be removed by a conformal transformation, and the conformally transformed metric is

$$\tilde{ds}^2 = -d\tau^2 + d\theta^2 + sin^2\theta d\Omega_{d-2}^2. \tag{24}$$

The original global coordinate ρ was defined for $\rho \geq 0$, and therefore $0 \leq \theta < \frac{\pi}{2}$. However, as previously seen in the Minkowski case, the metric is now regular for $\theta = \frac{\pi}{2}$ and the space can be conformally compactified to include this point - we can now see that AdS_d is conformal to one-half of the Einstein Static Universe. Just like flat space, if we can conformally compactify a spacetime to have the same conformal structure as AdS then we call it Asymptotically AdS, or $AAdS$. If we take the conformally compactified AdS_d to the boundary at $\theta = \frac{\pi}{2}$ we see that we have the topology $\mathbb{R} \times S^{d-2}$, which is the same as the conformally compactified Minkowski Space on $\mathbb{R}^{1,d-2}$. This is one of the key points of the AdS/CFT correspondence - when we take AdS_5 to the boundary, it looks like Minowski in $(3+1)$-dimensions, where the conformal field theory, $\mathcal{N} = 4$ Super Yang-Mills, lives.

4.1.4 Poincaré Coordinates

Another useful paramaterisation of the embedding of AdS_d is given by Poincaré coordinates, defined as the set $(\overrightarrow{x}, t, u)$ and $u > 0$. The coordinates cover half of the space, and are given by

$$\begin{aligned} X_0 &= \frac{1}{2u}(1 + u^2(R^2 + \overrightarrow{x}^2 - t^2)) \\ X_i &= Rux_i \\ X_d &= Rut \\ X_{d-1} &= \frac{1}{2u}(1 - u^2(R^2 - \overrightarrow{x}^2 + t^2)), \end{aligned}$$

with metric

$$ds^2 = R^2(\frac{du^2}{u^2} + u^2(-dt^2 + d\overrightarrow{x}^2)).$$

This can be brought into a slightly nicer form by equating $u = \frac{1}{z}$,

$$ds^2 = \frac{R^2}{z^2}(dz^2 + (-dt^2 + d\overrightarrow{x}^2)). \tag{25}$$

The conformal boundary is now at $z = 0$. This form of the metric is of particular use when studying the behaviour of the fields in the 'bulk' of AdS and their asymptotic behaviour. The boundary of the space lives at $u \to \infty$, corresponding to $z \to 0$. Obviously this creates a divergent metric - the volume of the space is infinite, and thus suffers from IR divergences. As we will see, the AdS/CFT correspondence is a duality, and thus 'long' in one corresponds to 'short' in another. Through this we are able to express this IR divergence in terms of the UV entropy divergences that Super Yang-Mills suffers at arbitrarily small distances. Not only that, but renormalising one theory results in the renormalisation of the other.[26] This is the idea of *holographic* renormalisation.

5 String Theory and D-branes

In the formulation of the AdS/CFT correspondence we consider Type IIB String Theory. The theory is one of closed oriented superstrings that contain the theory of gravity in the bulk of spacetime, and open strings ending on dynamical hypersurfaces called D-branes. D-branes are a crucial element, as their presence within the theory can distort the spacetime to look like AdS nearby. They also contain gauge theories on their worldvolume, providing the link between both sides of the correspondence. To study the massive degrees of freedom within this theory, we use Wigner's classification. As we'll be dealing with the low-energy dynamics of the theory, we consider here just the massless representations, those of the Type IIB supergravity theory, characterised by the little group in 10 dimensions, $SO(8)$.

5.1 Spectroscopy - Closed Strings

We begin by looking at the closed strings of Type IIB theory. Although the spectrum cannot be obtained directly from dimensional reduction, it is still instructive to do so; finding the Type IIA spectrum, and using tensor products to deduce the spectrum of Type IIB.

5.1.1 Supergravity in 11 Dimensions

In 11 dimensions the supergravity multiplet consists of

- a graviton, $g_{\mu\nu}$: rank 2 symmetric traceless tensor with $\frac{10.9}{2} - 1 = 44$ degrees of freedom
- a gravitino, ψ^{α}_{μ} : Majorna spin-$\frac{3}{2}$ vector-spinor with $8.2^4 = 128$ degrees of freedom
- third-rank antisymmetric tensor (three-form) $C_{\nu\mu\rho}$ with $\frac{9.8.7}{3.2} = 84$ degrees of freedom

The graviton and three-form make up the required 128 bosonic degrees of freedom, and the gravitino the 128 fermionic degrees of freedom, satisfying the requirement from supersymmetry that, within a given multiplet, there should be an equal number of bosonic and fermionic degrees of freedom. To classify these irreps, we look to the little group $SO(9)$ represented by the Dynkin diagram B_4.

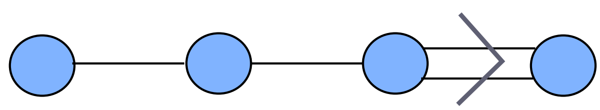

Figure 2: Dynkin diagram B_4, representing the Lie algebra of $SO(9)$

This diagram is classified by four labels $[n_1, n_2, n_3, n_4]$. In the highest weight representation, which is chosen because the weights are non-degenerate and thus uniquely define the irrep, there are four basic representations of $SO(9)$, given by

1. $[1000]_9$ vector representation, of dimension 9.
2. $[0100]_9$ second-rank antisymmetric tensor representation, of dimension $\binom{9}{2} = 36$
3. $[0010]_9$ third-rank antisymmetric tensor representation, of dimension $\binom{9}{3} = 84$
4. $[0001]_9$ spinor representation, of dimension $2^4 = 16$

Using the basic representations, and Weyl's dimensional formula, we can represent the graviton by $[2000]_9$, the gravitino by $[1001]_9$ (with one vector and one spinor label), and the three-form by $[0010]_9$.

5.1.2 Dimensional Reduction to Type IIA SUGRA

Now the 11 dimensional spectrum has been classified, we can use branching rules for the irreps to dimensionally reduce to Type IIA SUGRA in 10 dimensions. To classify the degrees of freedom in 10 dimensions, we look at the algebra for $SO(8)$, represented by the Dynkin diagram D_4

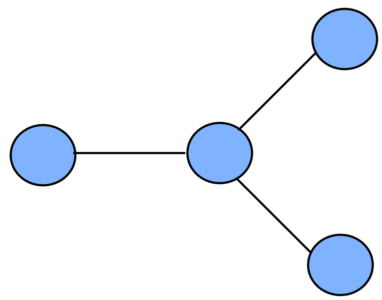

Figure 2: Dynkin diagram D_4, representing the Lie algebra of $SO(8)$

Here, the basic representations are given by

1. $[1000]_8$ vector representation: dimension $= 8$
2. $[0100]_8$ adjoint representation: dimension $\binom{8}{2} = 28$
3. $[0010]_8$ spinor s representation: dimension $2^3 = 8$
4. $[0001]_8$ spinor s' representation: dimension $2^3 = 8$

The s and s' spinor representations are of different chiralities, and have the same dimension as the vector representation. This is unique to D_4, the most symmetrical Dynkin diagram, and is known as triality. These three vector spaces can be permuted by an S_3 permutation arbitrarily[10], which will be useful for the classification of irreps. With knowledge of the dimensionalities of the $SO(8)$ basic representations, we can now look at how $SO(9)$ embeds itself into $SO(8)$. For the basic representations, the branching rules give

- $[1000]_9 = [1000]_8 + [0000]_8 \quad \rightarrow \quad \underline{\mathbf{9}} = \underline{\mathbf{8}} + \underline{\mathbf{1}}$
- $[0100]_9 = [0100]_8 + [1000]_8 \quad \rightarrow \quad \underline{\mathbf{36}} = \underline{\mathbf{28}} + \underline{\mathbf{8}}$
- $[0010]_9 = [0010]_8 + [0100]_8 \quad \rightarrow \quad \underline{\mathbf{84}} = \underline{\mathbf{56}} + \underline{\mathbf{28}}$
- $[0001]_9 = [0010]_8 + [0001]_8 \quad \rightarrow \quad \underline{\mathbf{16}} = \underline{\mathbf{8}} + \underline{\mathbf{8}}'$

To find the closed string spectrum of Type IIA, we dimensionally reduce the 11D SUGRA spectrum - the three-form reduction is already given above by $[0010]_9$ branching rules. For the graviton $[2000]_9$ and gravitino $[1001]_9$

- $[2000]_9 = [2000]_8 + [1000]_8 + [0000]_8 \quad \rightarrow \quad \underline{\mathbf{44}} = \underline{\mathbf{35}} + \underline{\mathbf{8}} + \underline{\mathbf{1}}$
- $[1001]_9 = [1001]_8 + [1010]_8 + [0001]_8 + [0010]_8 \quad \rightarrow \quad \underline{\mathbf{128}} = \underline{\mathbf{56}} + \underline{\mathbf{56}}' + \underline{\mathbf{8}} + \underline{\mathbf{8}}'$

where we've distinguished between the two chiralities. Using the above, the spectrum of type IIA supergravity is

$g_{\mu\nu}$	$B_{\mu\nu}$	ϕ	$A_{1\mu}$	$A_{3\mu\nu\rho}$	$\psi^{\mu}_{\pm\alpha}$	$\lambda_{\pm\alpha}$
$\underline{\mathbf{44}}$	$\underline{\mathbf{28}}$	$\underline{\mathbf{1}}$	$\underline{\mathbf{8}}$	$\underline{\mathbf{56}}$	$\underline{\mathbf{56}}, \underline{\mathbf{56}}'$	$\underline{\mathbf{8}}, \underline{\mathbf{8}}'$

This theory contains fermions of both chiralities, making Type IIA a non-chiral theory. To determine the Type IIB spectrum, we begin by looking at tensor products.

5.1.3 Tensor Products and Type IIB Spectrum

A complete decription of tensor products is given by Slansky. Useful for our purposes is that for two highest weights a_i and b_i, the tensor product leads with the weight $a_i + b_i$[25]. Sometimes the rules may not illuminate the entire tensor product, but we can often use informed guessing to complete this, such as using the knowledge of the degrees of freedom of the basic and composite representations. We are interested in $SO(8)$, and first consider $[1000]_8 \otimes [1000]_8$. Using the aforementioned rule, the leading term gives the graviton

$$[1000]_8 \otimes [1000]_8 = [2000]_8 + \dots$$

Comparing the degrees of freedom of both sides give $\underline{\mathbf{8}} \times \underline{\mathbf{8}} = \underline{\mathbf{35}}$, indicating that some are missing, so the tensor product must give further irreps. The product of two vectors gives a matrix, which can be broken down into a second-rank antisymmetric tensor, symmetric-traceless, and the trace. Therefore we postulate that this tensor product gives

$$[1000]_8 \otimes [1000]_8 = [2000]_8 + [0100]_8 + [0000]_8$$

and the degrees of freedom now read $\underline{\mathbf{8}}_v \times \underline{\mathbf{8}}_v = \underline{\mathbf{35}} + \underline{\mathbf{28}} + \underline{\mathbf{1}}$, agreeing. Next we consider the tensor product of a vector and a spinor $[1000]_8 \otimes [0001]_8$. The leading term must be a vector-spinor object, and the only one of this kind at our disposal is the gravitino $[1001]_8$ - comparing the degrees of freedom for the leading term gives $\underline{\mathbf{8}}_v \times \underline{\mathbf{8}}_s = \underline{\mathbf{56}}$, and thus we are missing an $\underline{\mathbf{8}}$. However, for $SO(8)$ three basic representations have this dimension; which representation do we choose? Using triality, we find that the result is a spinor of the opposite chirality, $\underline{\mathbf{8}}_{s'}$.

$$[1000]_8 \otimes [0001]_8 = [1001]_8 + [0010]_8 \quad \underline{\mathbf{8}}_v \times \underline{\mathbf{8}}_s = \underline{\mathbf{56}} + \underline{\mathbf{8}}_{s'}$$

Finally, we consider the tensor product of two spinors of opposite chirality $[0010]_8 \otimes [0001]_8$. The leading term is $[0011]_8$, but looking at the degrees of freedom we find that we are missing an $\underline{\mathbf{8}}$ - again using triality we can determine that this is the $\underline{\mathbf{8}}_v$ of the vector representation:

$$[0010]_8 \otimes [0001]_8 = [0011]_8 + [1000]_8 \quad \underline{\mathbf{8}}_s \times \underline{\mathbf{8}}_{s'} = \underline{\mathbf{56}} + \underline{\mathbf{8}}_v$$

By looking at these tensor products we are able to find all of the simple and composite irreps of Type IIA SUGRA, and we may express them in a factorised form as

$$\left([1000]_8 + [0010]_8\right)\left([1000]_8 + [0001]_8\right)$$

Spinors of both chiralities are used for this non-chiral theory - Type IIB is chiral, so it is natural to consider the chiral tensor product factorisation $\left([1000]_8 + [0001]_8\right)^2$, which gives us the Type IIB spectrum

- $[2000]_8$: graviton
- $[0100]_8$: NSNS two-form
- $[0000]_8$: dilaton
- $2[1001]_8$: gravitini
- $2[0010]_8$: spinors
- $[0000]_8$: RR zero-form
- $[0100]_8$: RR two-form
- $[0002]_8$: self-dual four-form

5.1.4 Brane Spectroscopy

By considering an extension of electrostatics and charge conservation, we can see how branes enter into the theory. For the AdS/CFT Correspondence, we will be particularly interested in Dp-branes, which carry charges under the Ramond-Ramond forms[24]. Initially, one can consider electromagnetism in $(3+1)$-dimensions. We have a gauge field, which is a one-form $A \equiv A_\mu dx^\mu$ with a two-form field strength $F \equiv F_{\mu\nu} dx^\mu \wedge dx^\nu$. Looking at the differential form of Maxwell's equation for an electric and magnetic source, we have

$$\begin{aligned} d * F &= \delta^{(3)} Q_e \\ dF &= \delta^{(3)} Q_m, \end{aligned}$$

representing an electric and magnetic point charge represectively, localised in three spatial dimensions. The Hodge star $*$ takes an r-form to a $(D-r)$-form, and the exterior derivative takes an r-form to an $(r+1)$-form. For the electric source equation example above, $*F$ is a two form, and taking the exterior derivative gives a three-form $\delta^{(3)}$ on the RHS.

Type IIB theory has an NS-NS two-form, as well as an RR zero-form, two-form and self-dual four-form. Extending the $(3+1)$ Maxwell case, where $D = 10$, we consider the NS-NS two-form $B_{\mu\nu}$, with corresponding field strength $H^{(3)} = dB \quad \rightarrow \quad d * H^{(3)} = \delta^{(8)} Q_e$, representing an electric charge localised in 8-dimensions and spanning one dimension. This is the fundamental string, denoted F_1. Similarly $dH^{(3)} = \delta^{(4)} Q_m$, which is a magnetic charge localised in 4 spatial dimensions and spanning five dimensions, which we call the NS5-brane. We can take the RR forms also :

$$C^{(0)} \begin{cases} d * F^{(1)} = \delta^{(10)} Q_e & \text{D(-1)-brane, instanton} \\ dF^{(1)} = \delta^{(2)} Q_m & \text{D7-brane} \end{cases}$$

$$C^{(2)} \begin{cases} d * F^{(3)} = \delta^{(8)} Q_e & \text{D1-brane} \\ dF^{(3)} = \delta^{(4)} Q_m & \text{D5-brane} \end{cases}$$

$$C^{(4)} \begin{cases} d * F^{(5)} = \delta^{(6)} Q_e & \text{D3-brane} \\ dF^{(5)} = \delta^{(6)} Q_m & \text{D3-brane.} \end{cases}$$

The self-duality of the four-form results in the D3-brane carrying both electric and magnetic charge; it is a dyonic object. This is the brane spectrum of Type IIB theory.

5.2 Open String Spectrum and D-branes

We have obtained the massless spectrum of the closed strings, which give rise to Type IIB supergravity. However, the correspondence involves a gauge theory living on a D-brane, and we do not see gauge fields in the supergravity theory - where do these additional degrees of freedom come from? These can be found upon the quantisation of the open strings propagating in the theory[28]. To be a theory relating to nature, the open superstrings should contain both fermions ψ, $\bar{\psi}$ and bosons X, and the action describing the dynamics (given in the superconformal gauge) is written as

$$S \sim \int d\sigma d\tau \left(\partial_a X^\mu \partial_a X^\mu - i \bar{\psi}^\mu \rho^a \partial_a \psi^\mu \right),$$

whose equations of motion yield the familiar Klein-Gordon and Dirac equations in two dimensions. As the open strings have endpoints, they are subject to boundary conditions, both bosonic and fermionic for the superstring theory. Varying the action with repsect to X^μ yields the boundary condition

$$\delta S \sim \int dt \left[\delta X^\mu \partial_a X_\mu \right]_0^\pi,$$

which we require to vanish. Therefore, we can impose two boundary conditions

$$\begin{aligned}\partial_a X^\mu(0,t) &= 0 \quad \text{Neumann} \\ \delta X^\mu(0,t) &= 0 \quad \text{Dirichlet.}\end{aligned} \tag{26}$$

The Neumann boundary condition restricts the end points of the string to only move freely in a plane. The latter, the Dirichlet condition, is more interesting. It implies that the end points of the strings are fixed at some particular position in time. It is this hypersurface that we call the D-brane. We may also see the need for this object within the theory by looking at Gauss' Law for the open string propagating in ten-dimensional space

$$Q = \int_{S^7} \overrightarrow{E}.d\overrightarrow{a}.$$

Taking the integral over S^7 gives us the total charge contained. We can first consider the string ending at some point in spacetime. If we imagining moving the sphere along the string until falling off the endpoint, we see that the charge is not conserved. Therefore, an open string simply cannot end, it must end on an object, namely the D-brane. In addition to this, the point charge acts as a source for a $U(1)$ gauge theory on the worldvolume of the brane. There are also fermionic boundary conditions known as Neveu-Schwarz and Ramond boundary conditions, which are given in terms of the chirality of the fermions

$$\begin{aligned}\psi^\mu_-(0,t) &= \xi_1\psi^\mu_+(0,t) \\ \psi^\mu_-(\pi,t) &= \xi_2\psi^\mu_+(\pi,t),\end{aligned} \tag{27}$$

where $\xi_1, \xi_2 = \pm 1$. The Ramond boundary condition is periodic, and the Neveu-Schwarz is anti-periodic[17]. Upon quantisation, these boundary conditions give rise to half-integer (Neveu-Schwarz) and integer (Ramond) mode expansions, which lead to spacetime bosons and fermions respectively. Both of these sectors taken separately would of course not constitute a natural theory, and certain other issues arise, such as the existence of a tachyon. However, we can use the a process derived by Gliozzi-Scherk-Olive, known as the GSO projection, to combine fermions and bosons and eradicate the issues[6], leaving us with the massless string spectrum of

- a gauge field A_μ with 8 degrees of freedom, given by the $\mathbf{8_v}$ represenation of $SO(8)$
- a Weyl spinor, λ_α,which may either in the $\mathbf{8}_s$ or $\mathbf{8_{s'}}$ representations of $SO(8)$

Thus, the string carries 8 bosonic and 8 fermionic degrees of freedom, and gives rise to field that are seen in gauge theories. Infact, by imposing Dirichlet conditions on the open string in $9-p$ directions, we can describe the dynamics of the D-brane from the excitations of the open string. To do this, we consider Lorentz invariance in 10 dimensions, given by the group $SO(9,1)$. The presence of a D-brane breaks this invariance to

$$SO(9,1) \supset SO(p,1) \times SO(9-p)_R,$$

where $SO(p,1)$ is Lorentz on the Dp-brane and $SO(9-p)_R$ is the rotational symmetry in the $(9-p)$-dimensions transverse to the brane. By Goldstone's Theorem, for every broken generator there should be a corresponding massless mode, and so we should expect $(9-p)$ scalars. The massless degrees of freedom within the vector multiplet can be classified using the little group, which for 10 dimensional massless theory is $SO(8)$, and decomposes in the presence of a Dp-brane to

$$SO(8) \supset SO(p-1) \times SO(9-p)_R.$$

We can begin by considering the vector multiplet in $(9+1)$-dimensions, where no decomposition is required. The bosonic degrees of freedom are given by the gauge field, with $[1000]_8$, and the 8 fermionic degrees of freedom are given by the gaugino, $[0001]_8$. The D9-brane fills all of spacetime, and thus $SO(9,1)$ is preserved. As there are no broken generators, there are no additional massless scalars. For the AdS/CFT correspondence, we are interested in a gauge theory in $(3+1)$-dimensions, and so it is natural to consider $p=3$, a D3-brane. The little group is broken to

$$SO(8) \supset SO(2) \times SO(6),$$

where $SO(2)$ is the little group on the worldvolume of the brane, and $SO(6)_R \sim SU(4)_R$ is the R-symmetry group, which crucially is the same R-symmetry group as $\mathcal{N}=4$ SYM in $d=4$. We can use the branching rules from dimensional reduction to find how the $(9+1)$-dimensional vector multiplet embeds itself into the separate products, maintaining the 8 bosonic and fermionic degrees of freedom. Doing this, we find

$$\begin{aligned} [1000]_8 &= [100]_6 q^0 + [000]_6 (q^2 + q^{-2}) \\ [0001]_8 &= [010]_6 q^1 + [001]_6 q^{-1}. \end{aligned} \tag{28}$$

Using $SO(2) \sim U(1)$, this representation has been labelled by the Abelian charge. By looking at the $SO(6)$ product with the q^n, the way that the $U(1)$ fields transform under the R-symmetry is manifest. Thus, on the world volume of the D3-brane, there is:

- A $U(1)$ gauge field, transforming as a singlet under $SO(6)$
- 6 scalars, transforming in the $\underline{\mathbf{6}}$ of $SO(6)$
- 4 complex Weyl fermions (gauginos), in the $\underline{\mathbf{4}}$ and $\underline{\bar{\mathbf{4}}}$ of $SO(6)_R$

This is the massless $U(1)$ abelian gauge theory on the D3-brane. We are looking to realise higher gauge groups, in particular the $SU(N)$ gauge group of $\mathcal{N}=4$ Super Yang-Mills. Consider N parallel branes, poisitioned arbitrarily at $\vec{x}_i$, $i=1,\ldots,N$. With this configuration we can engineer different gauge theories. In the simplest case, we may have strings that begin and end on the same brane, each contributing a $U(1)$ gauge theory, with resultant $U(1)^N$ gauge group. However, we can also consider strings that stretch between two distinct branes. These are oriented, and one string has $U(1)\times U(1)$ charge $(1,-1)$, whilst the other has charge $(-1,1)$. The mass of the states is $m_n \sim \frac{n}{L} + TL$, where the tension $T = l_s^{-2}$, with l_s the string length[12]. We see that the zero mode mass goes as $m_0 \sim TL$, so as the strings are brought to be coincidental, $L \to 0$, these modes become massless. For N coincidental branes, there are N^2 massless multiplets, giving the enhanced symmetry of $U(N)$. As we're interested in the dynamics of the gauge theory, we can ignore the $U(1)$ symmetry that corresponds to the position of the branes in spacetime, and thus this configuration gives rise to a $U(N)/U(1) \sim SU(N)$ gauge theory.

Considering the bosonic part of the theory, our massless excitations are the gauge field A_μ and scalars ϕ^I (in the $\underline{6}$ of $SO(6)$), both transforming in the adjoint representation. The effective action for an observer on the brane is given by

$$S \sim \int_{WV} d^4\xi Tr\left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}D_a\phi^I D_b\phi^I - \frac{1}{4}\sum_{I\neq J} Tr[\phi^I,\phi^J]^2\right), \tag{29}$$

where the first two terms are the kinetic terms of the gauge and scalar fields respectively, and the third term is the scalar potential. The integral is taken over the worldvolume of the brane, parameterised by coordinates ξ. We can extend this to include the fermionic sector, and thus find that the low-energy dynamics of N parallel D3-branes is given exactly by the $\mathcal{N} = 4$ Super Yang-Mills theory in $d = 4$ with $SU(N)$ gauge group.

6 Black Holes and p-branes

In the previous section we looked at D-branes as objects within string theory that carry the Ramond-Ramond charges, harbouring a gauge theory on their worldvolume. Below we are led to the second viewpoint, where D-branes are seen as solutions to the supergravity equations, deforming the geometry around them into AdS space. Both analyses from the crux of Malcadena's conjecture. It is natural to begin looking at the simplest example of a brane - the black hole - before extending to p-spatial dimensions, known as a p-brane. We consider the Schwarzschild black hole, which is non-rotating and uncharged.

6.1 Schwarzschild Metric

The Schwarzschild metric is a solution to Einstein's field equations in a vacuum, $R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 0$. It describes the behaviour of a gravitational field outside some spherically symmetric mass. The metric is given by[9]

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega_2^2, \tag{30}$$

where r is a radial coordinate measured from the origin at $r = 0$, and $d\Omega_2^2 = d\theta^2 + sin\theta d\phi^2$ is the spatial metric for a two-sphere. Birkhoff's theorem states that the Schwarzschild solution is the unique spherically symmetric solution to the Einstein equations. We can note that the metric has no explicit dependence on t and admits a killing vector $\xi = \partial_t$. Consequently, this means that any spherically symmetric solution must be static, or independent of time. One interesting consequence of Birkhoff's theorem is that the solution does not emit gravitational radiation[14].
The metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -(1-\frac{2M}{r}) & 0 & 0 & 0 \\ 0 & (1-\frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix}$$

It looks as though something funny may happen at $r = 2M$, indeed we find that at this radius the metric is not invertible and blows up - this is called the *event horizon* of the black hole. In order to try and understand the physicality of this, we can consider a particle on the surface of a ball of pressureless dust which is collapsing from rest at a radius R_{max}. The collapse is radially inwards ($d\theta = d\phi = 0$) and is spherically symmetric, so its exterior can be described by the Schwarzschild metric (the interior of the ball may have more complicated dynamics, but we do not consider them here). How long does it take for the particle to fall to a radius of $r = 2M$? The total time is expressed as

$$T_{2M} = \int_0^{t_{2M}} dt.$$

To evaluate this, we consider the action of a massive particle

$$S = \int d\tau \mathcal{L} = \int d\tau (-\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - m^2),$$

where τ is the proper time. Using the Euler-Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(\frac{dt}{d\tau})}\right) - \frac{\partial \mathcal{L}}{\partial t} = 0,$$

we find

$$\frac{d}{d\tau}\left(\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)\right) = 0.$$

The result defines a constant of motion along the worldline of the particle

$$\epsilon = \left(1 - \frac{2M}{r}\right))d\tau.$$

The initial condition for the particle was to fall from rest, $\frac{dR_{max}}{d\tau} = 0$, and for a massive particle we have $ds^2 = -d\tau^2$

$$\begin{aligned} \rightarrow \quad \left(\tfrac{dt}{d\tau}\right)^2 (1 - \tfrac{2M}{R_{max}}) &= 1 \\ 1 - \epsilon^2 = \tfrac{2M}{R_{max}}. \end{aligned}$$

Using these and solving the equations of motion for R, we find that the total time taken for a particle to fall to a radius of $2M$ is

$$T = \int_0^{t_{2M}} dt = -\epsilon \int_{R_{max}}^{2M} \frac{dR}{(1 - \frac{2M}{r})}(\frac{2M}{r} - \frac{2M}{R_{max}})^{-1/2}.$$

So as $r \rightarrow 2M$, the total time diverges and it seems as though the particle takes an infinite amount of time to reach the event horizon! Of course, we know this is not the case - so what is happening? It turns out that this is due to the choice of coordinates, t is a bad coordinate at $r = 2M$. Out towards $r \rightarrow \infty$, the metric is asymptotically flat, and t acts as though it measures what we call time. However, as we approach the horizon, this is no longer the case - coordinates have no physical meaning, they are just a parameterisation of spacetime. We can define a new coordinate r_* via the relation

$$dr_*^2 = (1 - \frac{2M}{r})^{-2} dr^2.$$

And go into Eddington-Finkelstein coordinates once more

$$\begin{aligned} v &= t + r_* \\ u &= t - r_*. \end{aligned}$$

Choosing the ingoing coordinate v, the Schwarzschild metric in terms of (v, r, θ, ϕ) becomes

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2 d\Omega_2^2. \tag{31}$$

If we do the same analysis as before, taking the radius to $r = 2M$, the (v, r) part of the metric reads

$$g_{\mu\nu} \mid_{r=2M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is invertible. Therefore the metric is perfectly regular at $r = 2M$; nothing special happens. This is an example of a coordinate singularity, where the singularity of the metric can be remedied by moving into another coordinate system. This is in contrast to a curvature singularity, which will exist in all coordinate bases, and has physical meaning within the spacetime. In the Schwarzschild example, a curvature singularity exists at $r = 0$; it is here that coordinate-invariant measures such as the Kretschmann scalar (the square of the Riemann tensor) diverge[2].

6.2 Reissner-Nordström Metric

In this example, we consider gravity coupled to an electric field, whose dynamics can be described by the Einstein-Maxwell action

$$S = \frac{1}{16\pi G} \int \sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu})d^4x, \tag{32}$$

where R is the Ricci scalar and $F_{\mu\nu}$ is the electromagnetic field strength. The equations of motion here also admit a spherically symmetric solution, describing a non-rotating black hole parameterised by a source charge Q and mass M. This solution is known as the Reissner-Nordström metric

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2 d\Omega_2^2.$$

In reality, these black holes are unlikely to exist - the universe is approximately neutral, so if a black hole were to become charged, it would quickly neutralise by interacting with the matter surrounding it. However, as we will see, it is instructive to study this metric in order to aid our understanding of p-branes in string theory. The coefficient

$$(1 - \frac{2M}{r} + \frac{Q}{r^2})$$

is a polynomial in r and may be factorised in the form $\Delta(r) = (r - r_-)(r - r_+)$. Solving the quadratic equation $r^2 - 2Mr + Q = 0$ gives us two roots

$$r = r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{33}$$

As there are two parameters, Q and M, we have three situations:

1. $M < Q$: this is the superextremal case; the roots become imaginary and there are no real solutions. Thus, there is no event horizon surrounding the black hole and we have a naked singularity, observable to the outside universe, at $r = 0$. However, it is strongly believed that naked singularities do not exist in nature, proposed in the conjecture known as Cosmic Censorship by Penrose[23]. Indeed, if they were to exist, then they would be apparent for us to observe, which has not been the case.

2. $M > Q$: this is the subextremal case. We have two real roots and two separate event horizons; r_+ is the outer horizon, r_- is the inner horizon. Although not physical, the subextremal case has a few interesting properties - the inner horizon where an observer falling over may see the infinite history of the universe, and internal infinities.

3. $M = Q$: this is the extremal case, and the most interesting for the purposes of this thesis. Here, both horizons converge into a single horizon at $r = M$.

6.3 Extremal Reissner-Nordström Metric

This metric has a coordinate singularity at $r = M$. We can make a change of coordinates, defining $r = M + \bar{r}$, such that

$$\left(1 - \frac{M}{r}\right) = \left(1 - \frac{M}{M+\bar{r}}\right) = \left(\frac{M+\bar{r}}{\bar{r}}\right)^{-1} = \left(1 + \frac{M}{\bar{r}}\right)^{-1},$$

with the metric taking on the form

$$ds^2 = -\left(1 + \frac{M}{\bar{r}}\right)^{-2} dt^2 + \left(1 + \frac{M}{\bar{r}}\right)^2 d\bar{r}^2 + \bar{r}^2 \left(1 + \frac{M}{\bar{r}}\right)^2 d\Omega_2^2.$$

From here we are able to define an Harmonic function (satisfying Lalpace's equation $\Delta H = 0$) $H = (1 + \frac{M}{\bar{r}})$, giving the metric

$$ds^2 = -H^{-2}dt^2 + H^2(d\bar{r}^2 + \bar{r}^2 d\Omega_2^2).$$

Looking at long distance, as $\bar{r} \to \infty$, the metric becomes asymptotically flat, suggesting that gravity is very weak. We can also look at the spacetime around $\bar{r} = 0$, which corresponds to the region around the event horizon of the extremal Reissner-Nordström black hole, and as such is known as *near-horizon*. In this regime, the metric takes the form

$$ds^2 \to -(\frac{\bar{r}}{M})^2 dt^2 + \left(\frac{M}{\bar{r}}\right)^2 d\bar{r}^2 + M^2 d\Omega_2^2,$$

and, by making the substitution $z = \frac{M^2}{r}$, it can be brought into a simpler form

$$ds^2 \mid_{\bar{r}\approx 0} = \frac{M^2}{z^2}(-dt^2 + dz^2) + M^2 d\Omega_2^2. \quad (34)$$

The near horizon geometry of the extremal Reisnner-Nordström black hole is precisely the product space $AdS_2 \times S^2$, both of radius M. Although innocuous-looking, this is an important result, as it is the first glimpse into the effect that the presence of a gravitational object has on the geometry of spacetime.

6.4 p-branes

We make an extension from the pointlike black hole to p-spatial dimensions. These are known as p-branes, and are solutions of 10-dimension supergravity equations, sourcing the graviton, dilaton and the Ramond-Ramond potentials in the 10-dimensional supergravity sprectra. The metric reads [15]

$$ds^2 = Z_p^{-\frac{1}{2}}(r)\left(-K(r)dt^2 + \sum_{i=1}^{p} dx_l^2\right) + Z_p^{\frac{1}{2}}(r)\left(\frac{dr^2}{K(r)} + r^2 d\Omega_{8-p}^2\right), \quad (35)$$

where [15]

- $Z_p(r) = 1 + \alpha_p \left(\frac{r_p}{r}\right)^{7-p}$
- $K(r) = 1 - (\frac{r_H}{r})^{7-p}$
- $\alpha_p = \sqrt{1 + \left(\frac{r_H^{7-p}}{2r_p^{7-p}}\right)^2} - \frac{r_H{}^{7-p}}{2r_p^{7-p}}$

p-branes span p-dimensions, but remain localised in the other $9-p$ dimensions transverse to the brane, admitting spherical symmetry. Because of this, we can parameterise the space in these dimensions with a raidal coordinate r and polar coordinates on the $8-p$ sphere. Observing the metric for the p-brane, the analogy to the black holes described above is obvious, and it is natural to ask whether there is an analogous definition for an 'event horizon' and coordinate singularity. The answer is yes: the event horizon, with a coordinate singularity, is positioned at the radius $r = r_H$, and the curvature singularity is once again at $r = 0$. We can take $r_H = 0$, giving the metric

$$ds^2 = H_p^{-\frac{1}{2}}(r)\left(-dt^2 + \sum_{i=1}^{p} dx_i^2\right) + H_p^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_{8-p}^2\right),$$

giving the *extremal* p-brane, analogous to the Reissner-Nordström metric. We define the harmonic function $H_p(r) = 1 + (\frac{r_p}{r})^{7-p}$ (as $\alpha_p = 1$). This also generalises to the situation where we have more than one brane, known as the multicentre solution[15]

$$H_p(r) = 1 + \sum_{i=1}^{N} \frac{r_p^{7-p}}{|\vec{r} - \vec{r_i}|^{7-p}}, \tag{36}$$

which represents N parallel branes situated at some arbitary position $\vec{r}_i$, each with charge N_i, such that the total charges sum to N. For the purposes of this investigation, the example that we would like to concentrate on is $p = 3$. For all other choices of p, the area of the $8-p$ sphere vanishes for $r = 0$, giving a singularity of zero area. However, for $p = 3$, the factor of r in the harmonic function cancels with the coefficient of the 5-sphere, leaving a finite area. For $p = 3$ we can explore the geometry of this solution near $r = 0$.

$$ds^2 \mid_{r\approx 0} = \frac{r^2}{r_3^2}(-dt^2 + d\vec{x}^2) + \frac{r_3^2}{r^2}(dr^2 + r^2 d\Omega_5^2).$$

Making an analogous substitution to the Reissner-Nordström case, $z = \frac{L^2}{r}$, gives the metric in Poincaré coordinates

$$ds^2 \mid_{r\approx 0} = \frac{L^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) + L^2 d\Omega_5^2. \tag{37}$$

The near horizon geometry is the product spacetime $AdS_5 \times S^5$, and will play a very important role within the correspondence!

7 AdS/CFT Correspondence

Finally, with all of the ingredients in place, we come to the correspondence. In the previous sections two objects were studied: the D-brane and the p-brane. In 1995, Polchinski showed that these two objects are infact the same - supergravity solutions correspond to the dynamical endpoints of strings.[24] This gives two viewpoints of D3-branes

- a configuration of N parallel D3-branes with strings stretched between. When they coincide, the low energy dynamics are described by $\mathcal{N} = 4$ Super Yang-Mills theory with $SU(N)$ gauge symmetry
- a configuration of N extremal D3-branes which are solutions of supergravity. The near-horizon geometry of these solutions is the product of spacetime $AdS_5 \times S^5$.

These viewpoints form the foundation of the conjecture, but we must consider the decoupling limits to make it apparent.

7.1 Decoupling Limit: Gauge Theory

We begin by considering Type IIB String Theory, with closed strings propgating in the bulk and open strings that end on the brane. We can therefore describe the dynamics of the theory with the action

$$S_{IIB} = S_{brane} + S_{bulk} + S_{int},$$

where:

- S_{brane} is the action defined on the $(3+1)$-dimensional world volume of the brane
- S_{bulk} is the action ten dimensional supergravity in the bulk
- S_{int} is the action describing the interaction between the bulk and the brane modes

In the low energy, $l_s \to 0$, limit we saw that the effective action on the brane is that of SYM gauge theory. Furthermore, the interaction term depends on the string lengt, $S_{int} \sim l_s^4$, and thus also vanishes[1]. The brane modes decouple from the bulk modes, leaving a SYM defined on the brane and free supergravity propagating in the bulk.

7.2 Decoupling Limit: Supergravity Theory

Next we consider the D-branes as supergravity solutions. The near horizon geometry can be described by $AdS_5 \times S^5$, and as $r \to \infty$, the geometry is asymptotically flat. The action of a relavistic massive particle is given by

$$S \sim \int \mathcal{L} d\tau \sim \int (g_{\nu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - m^2)^{\frac{1}{2}} d\tau.$$

Singling out time, the action can be extremised to find the Euler-Lagrange Equations

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{dt}{d\tau}\right)} \right) - \frac{\partial \mathcal{L}}{\partial t} = 0.$$

From Birkhoff's theorem and the extension to p-spatial branes, the supergravity solutions are static, and the second term vanishes, giving

$$\frac{d}{d\tau} \left(-2 H_p^{-\frac{1}{4}} \frac{dt}{d\tau} \right) = 0,$$

and thus, as we can determine a constant of motion, we see the proper time interval is related to dt by

$$d\tau = H_p^{-\frac{1}{4}} dt.$$

Suppose an observer sat at constant radius r_1 emits a photon to another observer at $r = \infty$

$$\begin{aligned} H_p(r_1) &= 1 + \left(\frac{r_p}{r_1}\right)^{7-p} \\ H_p(r = \infty) &= 1 + \left(\frac{r_p}{\infty}\right)^{7-p} = 1. \end{aligned}$$

The proper time intervals are $d\tau_1 = H_p^{-\frac{1}{4}}(r_1) dt$ and $d\tau_\infty = dt$. The frequency of the photons are inversely related to the proper time, and the redshift can be calculated by taking the ratio of frequencies between both observers

$$\frac{d\tau_1}{d\tau_\infty} = \frac{\omega_\infty}{\omega_1}$$

$$\begin{aligned} H_p^{-\frac{1}{4}}\omega_1 &= \omega_\infty \\ H_p^{-\frac{1}{4}}E_1 &= E_\infty, \end{aligned}$$

as $E \sim \omega$. Therefore we see that the photon has been redshifted by a factor of $H_p^{-\frac{1}{4}}$. As $r_1 \to 0$, the more the photon redshifts and the lower the energy to an observer at infinity. This is due to the gravitational potential well, analogous to approaching the event horizon of a black hole. The dynamics in the throat, where the spacetime is $AdS_5 \times S^5$, decouple from the dynamics in the bulk, where we also have a low energy regime (spacetime is asymptotically flat)[22], and the supergravity is free.
Both scenarios contain supergravity decoupled in flat space. Therefore, we are led to Malcadena's conjecture[16]:

Type IIB superstring theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ Super-Yang Mills theory in $(3+1)$-dimensions.

7.3 Limits and Validity

Now a relation between the gauge and string couplings has been made, the validity of the correspondence can be explored. For field theories, we can use perturbative analysis when the couplings are weak. For Super Yang-Mills, this corresponds to

$$g_{YM}^2 N \sim g_s N \sim \frac{L^4}{l_s^4} \ll 1. \tag{38}$$

On the other hand, supergravity only becomes reliable when the radius of curvature is greater than the string length

$$\frac{L^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1. \tag{39}$$

For supergravity to be valid, we typically require N to be large. There seems to be a complication - the two sides of the correspondence are valid in completely different regimes. This is the reason why the correspondence is called a duality - it relates the strong coupling of one theory to the weak coupling of the other, which in turn makes it very difficult to prove.
There are three forms of the correspondence with varying degrees of strength. These are:

- weakest form: valid for large $g_s N$, thus valid for the supergravity approximation. However, may not be valid for the full string theory away from this limit.
- mid-strength form: here we take the 't Hooft limit, keeping $\lambda = g_{YM}^2 N$ fixed, but taking $N \to \infty$ and $g_s \to 0$
- strongest form: valid for all g_s and N.

7.4 The Field-Operator Correspondence

For a duality to exist between two theories, there should be a one-to-one correspondence and mapping between their observables: fields, operators, and correlation functions. In the previous section we saw that the topological expansion derived by 't Hooft allowed us to relate the gauge theory coupling constant to that of the strings, $g_{YM}^2 \sim g_s$. In string theory, the string coupling is related to the dilaton field by

$$g_s = e^{\langle \phi \rangle}, \tag{40}$$

where $\langle \phi \rangle$ denotes the expectation value of the dilaton, which is set by the boundary conditions given at infinity[1]. Therefore, if the coupling strength of the gauge theory is changed, this will change the string theory boundary conditions for the dilation at the infinite boundary of space. For string theory in AdS_5, we see that the conformal boundary is that of a 4-dimensional Minkowski space, and can introduce a coupling of the CFT that lives on the boundary

$$\int d^4x \phi_0(x) \mathcal{O}, \tag{41}$$

where $\phi_0(x)$ is the source for the CFT operator $\mathcal{O}$. Because the field is fully determined by its boundary conditions, we want to ensure that the value of the bulk field tends towards $\phi_0(x)$ as it approaches the boundary

$$\phi(z, \overrightarrow{x}) \mid_{z=0} = \phi_0(\overrightarrow{x}).$$

On the CFT side of the correspondence, we have a source-operator relation, and would like to compute the correlation functions for distinct points $\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$. Witten extended this to defining a generating functional $\langle exp(\int_{S^4} \phi_0 d^4x \rangle_{CFT}$ and proposing the precise correspondence[31]

$$\langle exp(\int_{S^4} \phi_0 d^4x) \rangle_{CFT} = \mathcal{Z}_S \left[\phi(z, \overrightarrow{x}) \mid_{z=0} = \phi(\overrightarrow{x}) \right]. \tag{42}$$

The right hand side is the classical supergravity partition function, which may be written as

$$\mathcal{Z}_S \left[\phi(z, \overrightarrow{x}) \mid_{z=0} = \phi(\overrightarrow{x}) \right] = exp(-S_{SUGRA}(\phi)).$$

The Witten presciption will be used in the sections below, where we look at the structure of the propagators connecting the bulk dynamics to the boundary, and from this determine the correlation functions of the Super Yang-Mills theory

7.5 Wave Equation in AdS Space

The string theory side of the correspodence has a bulk-boundary relation, and we must understand how to express the dynamics of the scalar field in the bulk in terms of the boundary operator. We want to solve the wave equation in AdS_5, i.e. the Klein-Gordon equation for a scalar field

$$\left(\Box - m^2 \right) \phi = 0. \tag{43}$$

To do this, we make use of the metric given in Poincaré coordinates

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{ij} dx^i dx^j).$$

The boundary in these coordinates is given as $z \to 0$, which corresponds to the radial coordinate $r \to \infty$, thus, the gauge theory 'lives at infinity'. The metric in these coordinates is given as

$$g_{\mu\nu} = \frac{L^2}{z^2}(1, -1, 1, \dots, 1) \quad g^{\mu\nu} = \frac{z^2}{L^2}(1, -1, 1, \dots, 1).$$

The d'Alembertian

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu)$$

$$\begin{aligned} &= \tfrac{z^n}{L^n}\left[\partial_z\left(\tfrac{z^2}{L^2}\tfrac{L^n}{z^n}\partial_z\right) + \tfrac{L^n}{z^n}\tfrac{z^2}{L^2}g^{ij}\partial_i\partial_j\right] \\ &= \tfrac{z^n}{L^n}\left[\partial_z\left(\tfrac{L^{n-2}}{z^{n-2}}\partial_z\right) + \tfrac{L^{n-2}}{z^{n-2}}\eta^{ij}\partial_i\partial_j\right], \end{aligned}$$

where $g = detg_{\mu\nu} = -\frac{L^{2n}}{z^{2n}}$ and the (i,j) components are just the flat-space metric. Taking the Leibniz rule

$$\begin{aligned} &= \tfrac{z^n}{L^n}\left[\tfrac{L^{n-2}}{z^{n-2}}\partial_z^2 - (n-2)\tfrac{L^{n-2}}{z^{n-3}}\partial_z + \tfrac{L^{n-2}}{z^{n-2}}\eta^{ij}\partial_i\partial_j\right] \\ &= \tfrac{z^2}{L^2}\left[\partial_z^2 - \tfrac{(n-2)}{z}\partial_z + \eta^{ij}\partial_i\partial_j\right]. \end{aligned}$$

We follow the method of Avery[3] and rotate into Euclidean space to study the behaviour of the field ϕ near the boundary, with d'Alembertian

$$\Box = \frac{z^2}{L^2}\left[\partial_z^2 - \frac{(n-2)}{z}\partial_z + \nabla_{n-1}^2\right] \tag{44}$$

and consider the ansatz for the solution

$$\phi(z,\overrightarrow{x}) = \frac{z^\Delta}{(z^2+\overrightarrow{x}^2)^\Delta} \tag{45}$$

$$\phi(z,\overrightarrow{x}) = \frac{z^\Delta}{(z^2+\overrightarrow{x}^2)^\Delta}$$

for some parameter Δ. We see

$$\begin{aligned} \partial_z\phi &= \frac{\Delta z^{\Delta-1}(z^2+\overrightarrow{x}^2)^\Delta - 2\Delta z(z^2+\overrightarrow{x}^2)^{\Delta-1}z^\Delta}{(z^2+\overrightarrow{x}^2)^{2\Delta}} \\ &= \frac{z^\Delta\Delta}{(z^2+\overrightarrow{x}^2)^\Delta}\left[z^{-1} - \frac{2z}{(z^2+\overrightarrow{x}^2)}\right] \\ &= \frac{\Delta\phi}{z}\left[\frac{\overrightarrow{x}^2-z^2}{\overrightarrow{x}^2+z^2}\right] \end{aligned}$$

and with similar analysis[3]

$$\begin{aligned} \nabla^2\phi &= \frac{\phi\Delta}{z^2(z^2+\overrightarrow{x}^2)}\left[2(2\Delta+3-n)z^2\overrightarrow{x}^2 - 2(n-1)z^4\right] \\ \partial_z^2\phi &= \frac{\phi\Delta}{z^2(z^2+\overrightarrow{x}^2)^2}\left[(\Delta-1)\overrightarrow{x}^4 - 2(\Delta+2)z^2\overrightarrow{x}^2 + (\Delta+1)z^4\right]. \end{aligned}$$

Placing this into the wave equation

$$\Box\phi = \frac{\Delta(\Delta+1-n)\phi}{L^2},$$

which, for AdS_5 gives

$$\begin{aligned}\Delta(\Delta-4)-m^2L^2 &= 0\\ \Delta(\Delta-4) &= m^2L^2.\end{aligned} \tag{46}$$

This equation is a polynomial in Δ, and for the massive wave equation in general AdS_n, the solutions are

$$\Delta\pm = \frac{(n-1)}{2} \pm \frac{1}{2}\sqrt{(n-1)^2+4m^2L^2}. \tag{47}$$

The aim is to study the dynamics of the fields in the bulk given some boundary conditions, employing the method of the Green's function, which characterises the response of the bulk in the presence of a point source. A simple example of this is within electrostatics, with a point charge at position $\overrightarrow{x}$ - what should the charge density be? Well, at any position $\overrightarrow{x}' \neq \overrightarrow{x}$, the charge density is zero, but at $\overrightarrow{x}$ we have a finite charge within an infinitely small point, giving an infinite density. Algebraically, we can write this as

$$\rho(\overrightarrow{x}) = \delta(\overrightarrow{x}' - \overrightarrow{x})Q.$$

Analogously, we want to look at the response of our scalar field in the presence of a point source, and find the Green's function $G(z, \overrightarrow{x}; \overrightarrow{x}')$ such that

$$(\Box - m^2)G(z, \overrightarrow{x}; \overrightarrow{x}') = \delta(\overrightarrow{x} - \overrightarrow{x}'). \tag{48}$$

Looking at the scalar field

$$\phi(z, \overrightarrow{x}) = \frac{z^\Delta}{(z^2+\overrightarrow{x}^2)^\Delta},$$

we see that for $\overrightarrow{x} \neq 0$, the field vanishes as we approach the boundary at $z \to 0$, and at $\overrightarrow{x} = 0$, the field diverges. This behaviour is suggestive of a delta function, but to observe whether this holds we must integrate over the $n-1$ directions and approach $z = 0$[3].

$$\int d^{n-1}x\phi = z^\Delta \int d^{n-1}x \frac{1}{(z^2+\overrightarrow{x}^2)^\Delta} = z^\Delta \Omega_{n-2} \int_0^\infty dr \frac{r^{n-2}}{(z^2+r^2)^\Delta},$$

Where Ω_{n-2} is the volume of an $(n-2)$-dimensional ball, with the standard result

$$\Omega_{n-2} = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}$$

$$\begin{aligned}\Longrightarrow \quad &= z^\Delta \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_0^\infty dr \frac{r^{n-2}}{(z^2+r^2)^\Delta}\\ &= z^\Delta \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_0^\infty (2rdr) \frac{\frac{1}{2}r^{n-3}}{(z^2+r^2)^\Delta}\\ &= z^\Delta \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \frac{1}{z^{2\Delta}} \int_0^\infty (2rdr) \frac{\frac{1}{2}r^{n-3}}{(1+\frac{r^2}{z^2})^\Delta}.\end{aligned}$$

Making a change of variables $t = r^2/z^2$

$$
\begin{aligned}
\implies &= z^{\Delta}\frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\frac{z^{n-1}}{2z^{2\Delta}}\int_0^\infty \frac{t^{n-1}}{(1+t)^{\Delta}}dt \\
&= z^{\Delta}\frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\frac{z^{n-1}}{2z^{2\Delta}}B(\frac{n-1}{2},\Delta-\frac{n-1}{2}) \\
&= \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\frac{z^{n-1}}{z^{\Delta}}\frac{\Gamma(\frac{n-1}{2})\Gamma(\Delta-\frac{n-1}{2})}{\Gamma(\Delta)} \\
&= \pi^{(n-1)/2}z^{n-1-\Delta}\frac{\Gamma(\Delta-\frac{n-1}{2})}{\Gamma(\Delta)},
\end{aligned}
$$

where we used the beta function

$$B(x,y)=\int_0^\infty \frac{t^{x-1}}{t^{x+y}}dt=\frac{B(x)B(y)}{B(x+y)}.$$

We denote the coefficient of z as a normalisation constant C_n, and eventually find that

$$\int dx^{n-1}\phi = C_n z^{n-1-\Delta}. \tag{49}$$

The scalar field doesn't quite behave as a delta-function, but $\phi z^{\Delta+1-n}$ does, as z approaches the boundary. The dynamics of the bulk field can be described in terms of the boundary field and some boundary-to-bulk propagator. We give the bulk field as

$$\phi(z,\overrightarrow{x})=\int d^{n-1}yK_{\Delta}(z,\overrightarrow{x};\overrightarrow{y})\phi_0(\overrightarrow{y}),$$

where

$$K_{\Delta}(z,\overrightarrow{x};\overrightarrow{y})=\frac{1}{C_n}\frac{z^{\Delta}}{(z^2+(\overrightarrow{x}-\overrightarrow{y})^2)^{\Delta}}, \tag{50}$$

such that the wave equation is solved and the boundary is approached as $K_{\Delta}(z,\overrightarrow{x};\overrightarrow{y})\mid_{z\to 0}= z^{n-1-\Delta}\delta(\overrightarrow{x}-\overrightarrow{y})$. This is the *bulk-to-boundary* propagator. It is this powerful tool which allows us to write the dynamics of the bulk in terms of the boundary for the two and three point functions. The solution to the wave equation had two roots

$$\Delta\pm=\frac{(n-1)}{2}\pm\frac{1}{2}\sqrt{(n-1)^2+4m^2L^2},$$

and the asymptotic behaviour near the boundary forces us to take the Δ_+ solution, so as we approach the boundary the field behaves like $\phi\mid_{z\to 0}\sim z^{4-\Delta_+}\phi(\overrightarrow{y})=z^{\Delta_-}\phi_0(\overrightarrow{y})$ We can isolate the behaviour that satisfies this within the neighbourhood of the boundary, such that

$$\phi(\overrightarrow{x},\epsilon)=\epsilon^{\Delta_-}\phi_0(\overrightarrow{y}), \tag{51}$$

where we eventually take the limit $\epsilon\to 0$ to reach the boundary. Because the dilaton field is dimensionless, this implies that the boundary value has mass dimension Δ_-. This boundary field also acts as a source for the action coupling term $\int d^4x\phi_0\mathcal{O}$, which must be dimension, implying that the operator $\mathcal{O}$ has a conformal weight $4-\Delta_-=\Delta_+$. In the next section we will look at the correlation functions for operators of this conformal weight, checking that they conform to the restrictions imposed by the high symmetry of the conformal group.

7.6 Two-Point and Three-Point Functions

7.6.1 Two-Point Functions

In the previous section we saw that a field in the bulk of AdS_5 can be described in terms of its boundary value ϕ_0 via the bulk-to-boundary propagator, $K_{\Delta_+}(z,\overrightarrow{x};\overrightarrow{y})$. This gives the bulk-to-boundary relation

$$\phi(z,\overrightarrow{x}) = \int d^4y K_{\Delta_+}(z,\overrightarrow{x};\overrightarrow{y})\phi_0(\overrightarrow{y}) = \frac{1}{C_n}\int d^4y \frac{z^{\Delta_+}}{(z^2+|\overrightarrow{x}-\overrightarrow{y}|^2)^{\Delta_+}}\phi_0(\overrightarrow{y}).$$

The dynamics of a free scalar field in AdS_5 are given by the SUGRA action

$$S_{SUGRA} = \frac{1}{2}\int \sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)d^4xdz.$$

This can be integrated by parts, and by imposing the on-shell wave equation $(\Box - m^2)\phi = 0$, only the boundary term remains

$$\sqrt{g}g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi) = \partial_\mu(\sqrt{g}g^{\mu\nu}\phi\partial_\nu\phi) - \phi\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) = \partial_\mu(\sqrt{g}g^{\mu\nu}\phi\partial_\nu\phi) - \phi\sqrt{g}\Box\phi$$

$$\begin{aligned} S_{SUGRA} &= \frac{1}{2}\int \partial_\mu(\sqrt{g}g^{\mu\nu}\phi\partial_\nu\phi)d^4xdz - \int \sqrt{g}\phi(\Box - m^2)\phi d^4xdz \\ &= \frac{1}{2}\int \partial_\mu(\sqrt{g}g^{\mu\nu}\phi\partial_\nu\phi)d^4xdz. \end{aligned}$$

Witten noticed a nice trick here [31], using the divergence theorem to express this in terms of a surface integral

$$= \lim_{z\to 0}\left(-\frac{1}{2}\int_B \sqrt{g}g^{zz}\phi\partial_z\phi d^4x\right), \tag{52}$$

with a normal vector whose components are only non-zero in the z direction, i.e. they're perpendicular to the boundary. Setting the curvature radius $L = 1$ for ease and in Poincaré coordinates, $g^{zz} = z^2$ and $\sqrt{g} = z^{-5}$ for AdS_5

$$\Longrightarrow \quad = \quad \lim_{z\to 0} - \frac{1}{2}\int_B z^{-3}\phi\partial_z\phi d^{4x}.$$

The behaviour of $\partial_z\phi$ goes as

$$\begin{aligned} \partial_z\phi(\overrightarrow{x},z;\overrightarrow{y}) &= \frac{1}{C_n}\int d^4y\partial_z\left[\frac{z^{\Delta_+}}{(z^2+|\overrightarrow{x}-\overrightarrow{y}|^2)^{\Delta_+}}\right]\phi_0(\overrightarrow{y}) \\ &= \frac{\Delta_+}{C_n}\int d^4y\left[\frac{z^{\Delta_+-1}(z^2+|\overrightarrow{x}-\overrightarrow{y}|^2)^{\Delta_+} - 2z^{\Delta_++1}(z^2+|\overrightarrow{x}-\overrightarrow{y}|^2)^{\Delta_+-1}}{(z^2+|\overrightarrow{x}-\overrightarrow{y}|^2)^{2\Delta_+}}\right]\phi_0(y) \\ &= \frac{\Delta_+}{C_n}\int d^4y\left[\frac{z^{\Delta_++1} + z^{\Delta_+-1}|\overrightarrow{x}-\overrightarrow{y}|^2 - 2z^{\Delta_++1}}{(z^2+|\overrightarrow{x}-\overrightarrow{y}|^2)^{\Delta_++1}}\right]\phi_0(\overrightarrow{y}), \end{aligned}$$

and recalling, as we approach the boundary, ϕ behaves as $\phi\mid_{z\to 0} = z^{\Delta_-}\phi_0$. We also find that the powers of z cancel, leaving

$$S_{SUGRA} = \frac{1}{2}\frac{\Delta_+}{C_n}\int d^4x d^4y \frac{\phi_0(\overrightarrow{x})\phi_0(\overrightarrow{y})}{|\overrightarrow{x} - \overrightarrow{y}|^{2\Delta_+}}. \tag{53}$$

From Witten's postulate, the correlation function is given by

$$\langle \mathcal{O}(\overrightarrow{x_1})\mathcal{O}(\overrightarrow{x_2})\rangle = \left[\left(-i\frac{\delta}{\delta\phi_0(\overrightarrow{x_1})}\right)\left(-i\frac{\delta}{\delta\phi_0(\overrightarrow{x_2})}\right)e^{-S_{SUGRA}}\right]|_{\phi_0=0},$$

which, of course, is the normal definition of the correlation function by taking the functional derivative of the generating functional with sources ϕ_0.

$$\begin{aligned}
&= \left(-\frac{\delta}{\delta\phi_0(\overrightarrow{x_2})}\right)\left[\frac{1}{2}\frac{\Delta_+}{C_n}\left(\frac{\phi_0(\overrightarrow{x_1})\delta(\overrightarrow{x_1}-\overrightarrow{x})}{|\overrightarrow{x}-\overrightarrow{y}|^{2\Delta_+}} + \frac{\phi_0(\overrightarrow{x_1})\delta(\overrightarrow{x_1}-\overrightarrow{y})}{|\overrightarrow{x}-\overrightarrow{y}|^{2\Delta_+}}\right)e^{-S_{SUGRA}}\right]|_{\phi_0=0} \\
&= \left[-\frac{1}{2}\frac{\Delta_+}{C_n}\left(\frac{\delta(\overrightarrow{x_1}-\overrightarrow{x})\delta(\overrightarrow{x_2}-\overrightarrow{y})}{|\overrightarrow{x}-\overrightarrow{y}|^{2\Delta_+}} + \frac{\delta(\overrightarrow{x_1}-\overrightarrow{y})\delta(\overrightarrow{x_2}-\overrightarrow{x})}{|\overrightarrow{x}-\overrightarrow{y}|^{2\Delta_+}}\right)\right] \\
\langle \mathcal{O}(\overrightarrow{x_1})\mathcal{O}(\overrightarrow{x_2})\rangle &\sim \frac{\Delta_+}{C_n}\frac{1}{|\overrightarrow{x_1}-\overrightarrow{x_2}|^{2\Delta_+}}.
\end{aligned} \tag{54}$$

7.6.2 Three-Point Functions

For three point functions, we look for a cubic term in the supergravity action

$$S_{SUGRA} = \int d^5x\sqrt{g}\left[\sum_i \frac{1}{2}(\partial\phi_i)^2 + \frac{1}{2}m_i^2\phi_i^2 + \lambda\phi_1\phi_2\phi_3\right],$$

where for each field there is the bulk-to-boundary relation

$$\phi(z,\overrightarrow{x}) = \int d^4x_i K_{\Delta_+}(z,\overrightarrow{x};\overrightarrow{x_i})\phi_0(\overrightarrow{x_i}),$$

and using the same methods as the two point function, the three point function is given by

$$\begin{aligned}
\langle \mathcal{O}(\overrightarrow{x_1})\mathcal{O}(\overrightarrow{x_2})\mathcal{O}(\overrightarrow{x_3})\rangle &= -\lambda\int d^5x\sqrt{g}K_{\Delta_+}(x;\overrightarrow{x_1})K_{\Delta_+}(x;\overrightarrow{x_2})K_{\Delta_+}(x;\overrightarrow{x_3}) \\
&= \frac{\lambda C_{123}}{|\overrightarrow{x_1}-\overrightarrow{x_2}|^{\Delta_1+\Delta_2-\Delta_3}|\overrightarrow{x_1}-\overrightarrow{x_3}|^{\Delta_1+\Delta_3-\Delta_2}|\overrightarrow{x_2}-\overrightarrow{x_3}|^{\Delta_2+\Delta_3-\Delta_1}},
\end{aligned} \tag{55}$$

for some constant C_{123}, whose value is determined by the evaluation of the x integral [1]. Comparing equations (55), (56) with the conformally restricted forms of the correlations functions in (17) and (18), we see that these correlations are exactly what we would expect for operators of conformal dimension Δ_+.

8 Conclusion

In this paper, we have seen how two seemingly unrelated theories, one with gravity and one strictly without, can be brought together to form the duality that is the AdS/CFT Correspondence. This particularly exciting example of a strong/weak duality allows one to probe the strong coupling of field theories, difficult to access otherwise, through a perturbative expansion in a gravity theory. From this we can learn many lessons in QCD, and indeed use the the correspondence to formulate the idea of holographic QCD, providing a systematic approach in describing the messier sides of the field theory, such as nuclear physics and heavy ion collisions. By bringing string theory back to its original roots, we had to use a number of components from vast expanses of physics, each one crucial in providing a comprehensive understanding of the conjecture. Throughout the paper, there were tantalising hints of a duality between AdS_5/CFT_4, including

- The isometry group of AdS_5 is the conformal group symmetry in four dimensions, $SO(2,4)$
- The conformal boundary of AdS_5 space is that of Minkowski in four dimenions, $\mathbb{R}^{1,3}$, suggesting this domain is the home for the conformal field theory
- 't Hooft's large N limit and idea to look at the topological structure of Feynman diagrams uncovered the relation between the Yang-Mills coupling g_{YM} and the string coupling, g_s

There was no safe leap across the chasm without the introduction of D3-branes, extended objects lying at the heart of AdS_5/CFT_4. By looking at these as endpoints of open strings and solutions to supergravity, we were able to put forward the conjecture that Type IIB string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ Super-Yang Mills theory in $(3+1)$-dimensions. The holographic principle allowed us to use the dynamics of the bulk in AdS_5 space to formulate the two- and three-point correlation functions through the boundary value of the field ϕ_0, which acted as a source for the conformal operators.

This, however, is only one particular example of the gauge/gravity duality and, through the use of holography, we are able to choose a variety of brane configurations from both string theory and M-theory to realise more exotic field theories. In doing so, we illuminate shrouded paths of the labyrinth and, step by step, strive to provide a correspondence describing the nature around us.

Acknowledgements

I would like to thank Professor Dan Waldram for all of his help both throughout this thesis and the last one. I have thoroughly enjoyed working with him, and he has made me want to be a physicist in the future. I would also like to thank my parents for their unending love and support, and all of my friends for being there for me, even in the darkest of times. Finally, thank you to the phyiscs gang, it has been the best.

References

[1] O. Aharony, S. Gubser, H. Malcadena, and Y. Ooguri. Large n field theories, string theory and gravity, arxiv:hep-th/9905111v3. 1999.

[2] A. Ashketar and V Petkov. *Springer Handbook of Spacetime*. 2014.

[3] S. Avery. Ads-cft correspondence. 2008.

[4] J. Bagger and J. Wess. *Supersymmetry and Supergravity*. Princeton Series in Physics, 1992.

[5] M. Bertolini. Lectures on supersymmetry, http://people.sissa.it/ bertmat/susycourse. 2014.

[6] Massimo Bianchi. Gliozzi-scherk-olive projection. In *Concise Encyclopedia of Supersymmetry*, pages 168–168. Springer Netherlands, 2003.

[7] Sidney Coleman and Jeffrey Mandula. All possible symmetries of the s matrix. *Phys. Rev.*, 159:1251–1256, Jul 1967.

[8] E. D'Hoker and D. Freedman. Supersymmetric gauge theories and the ads/cft correspondence, http://arxiv.org/pdf/hep-th/0201253v2. February 2002.

[9] F. Dowker. Black holes, lectures in msc in quantum fields and fundamental forces. 2014.

[10] W. Fulton and J. Harris. *Representation Theory, A First Course*. Springer, 2004.

[11] R. Haag, J. T. Łopuszański, and M. Sohnius. All possible generators of supersymmetries of the S-matrix. *Nuclear Physics B*, 88:257–274, March 1975.

[12] A. Hanany. String theory,lectures in msc in quantum fields and fundamental forces, 2014.

[13] D. Herr. Cft: Basic properties and examples. March 2013.

[14] G. T. Horowitz. *Black Holes in Higher Dimensions*. April 2012.

[15] C. Johnson. *D-Branes*. University Press, Cambridge, 2003.

[16] J. Malcadena. The large n limit of superconformal field theories and supergravity, http://arxiv.org/abs/hep-th/9711200v3. jan 1998.

[17] L. McAllister. Lectures on string theory, March 2010.

[18] McGreevy. Large n counting. 2008.

[19] H. Muller-Kirsten and A. Wiedemann. *Introduction to Supersymmetry*. World Scientific, 2010.

[20] M. Nakahara. *Geometry, Topology and Physics, Second Edition*. Taylor and Francis Group, LLC, 2003.

[21] Yu Nakayama. Scale invariance vs conformal invariance, arxiv:1302.0884v4. February 2014.

[22] H Nastase. Introduction to ads-cft. 2007.

[23] R. Penrose. Rivista del nuovo cim. numero speciale 1, 252. 1969.

[24] J. Polchinski. Dirichlet-branes and ramond-ramond charges, hep-th/9510017. nov 1995.

[25] R. Slansky. Group theory for unified model building. *Physics Reports*, 79:1–128, December 1981.

[26] L. Susskind and E. Witten. The holographic bound in anti-de sitter space, arxiv:hep-th/9805114v1. May 1998.

[27] G. 't Hooft. A planar diagram theory for strong interactions. *Nuclear Physics B*, 72:461–473, April 1974.

[28] D. Tong. String theory: University of cambridge part iii mathematical tripos, January 2009.

[29] P. Townsend. Black holes, arxiv:gr-qc/9707012v1. 1997.

[30] P. West. *Introduction to Supersymmetry and Supergravity*. World Scientific, 1990.

[31] E. Witten. Anti de sitter space and holography, hep-th/9802150. Apr 1998.